MATHEMATICS-I

Deepak Singh

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FOREWORD

Engineering has played a very significant role in the progress and expansion of mankind and society for centuries. Engineering ideas that originated in the Indian subcontinent have had a thoughtful impact on the world.

All India Council for Technical Education (AICTE) had always been at the forefront of assisting Technical students in every possible manner since its inception in 1987. The goal of AICTE has been to promote quality Technical Education and thereby take the industry to a greater heights and ultimately turn our dear motherland India into a Modern Developed Nation. It will not be inept to mention here that Engineers are the backbone of the modern society - better the engineers, better the industry, and better the industry, better the country.

NEP 2020 envisages education in regional languages to all, thereby ensuring that each and every student becomes capable and competent enough and is in a position to contribute towards the national growth and development.

One of the spheres where AICTE had been relentlessly working from last few years was to provide high-quality moderately priced books of International standard prepared in various regional languages to all it's Engineering students. These books are not only prepared keeping in mind it's easy language, real life examples, rich contents and but also the industry needs in this everyday changing world. These books are as per AICTE Model Curriculum of Engineering & Technology – 2018.

Eminent Professors from all over India with great knowledge and experience have written these books for the benefit of academic fraternity. AICTE is confident that these books with their rich contents will help technical students master the subjects with greater ease and quality.

AICTE appreciates the hard work of the original authors, coordinators and the translators for their endeavour in making these Engineering subjects more lucid.

(**Anil D. Sahasrabudhe**)

Acknowledgement

The author is grateful to AICTE for their meticulous planning and execution to publish the technical book for Diploma students.

I sincerely acknowledge the valuable contributions of the reviewer of the book Dr. Pradip Nandlal Joshi, for making it students' friendly and giving a better shape in an artistic manner.

This book is an outcome of various suggestions of AICTE members, experts and authors who shared their opinion and thoughts to further develop the engineering education in our country.

It is also with great honour that I state that this book is aligned to the AICTE Model Curriculum and in line with the guidelines of National Education Policy (NEP) -2020. Towards promoting education in regional languages, this book is being translated in scheduled Indian regional languages.

Acknowledgements are due to the contributors and different workers in this field whose published books, review articles, papers, photographs, footnotes, references and other valuable information enriched us at the time of writing the book.

Finally, I like to express my sincere thanks to the publishing house, M/s. Khanna Book Publishing Company Private Limited, New Delhi, whose entire team was always ready to cooperate on all the aspects of publishing to make it a wonderful experience.

Deepak Singh

Preface

The book titled "Mathematics-I" is an outcome of teaching Mathematics for Engineering students. The initiation of writing this book is to expose basic concepts of Mathematics for Diploma engineering students to the fundamentals of Mathematics as well as enable them to get an insight of the subject. Keeping in mind the purpose of wide coverage and also to provide essential supplementary information, Author included the topics recommended by AICTE, in a very systematic and logical manner throughout the book. Efforts have been made to explain the fundamental concepts of the subject in the simplest possible way.

During the process of preparation of the manuscript, I have considered the various standard text books and accordingly sections like create inquisitiveness, solved and supplementary problems etc., have developed. While preparing the different sections emphasis has also been laidon comprehensive synopsis of formulae for a quick revision of the basic principles. Apart from Tapping into areas of student interest, the author provides an ample supply of examples and rich exercises. Found in every Unit, realistic applications draw students into the discipline to help them to generalize the material and apply it to new and novel situations. To further spark student interest, meticulously drawn graphs and illustrations appear throughout the text.

In addition, besides some essential information for the users under the heading "Know More" the Author have clarified some essential basic information for further readings.

As far as the present book is concerned, "Mathematics-I: is meant to provide a thorough grounding in the subject on the topics covered. This part of the Mathematics-I will prepare students to apply the knowledge of Trigonometry, Calculus and Algebra to address the related aroused questions. The subject matters are presented in a constructivemanner.

The Author sincerely hope that the book will inspire the students to learn and discuss the ideas behind basic principles of Trigonometry, Calculus and Algebra to the develop solid basis of the subject. I would be thankful to all beneficial comments and suggestions which will contribute to the improvement of the future editions of the book. It gives me immense pleasure to place this book in the hands of the teachers and students. It was indeed a big pleasure to work on different aspects covering in the book.

Deepak Singh

Outcome Based Education

For the implementation of an outcome-based education the first requirement is to develop an outcome-based curriculum and incorporate an outcome-based assessment in the education system. By going through outcome-based assessments evaluators will be able to evaluate whether the students have achieved the outlined standard, specific and measurable outcomes. With the proper incorporation of tools of outcome-based education there will be a certain commitment to achieve a minimum standard for all learners without giving up at any level. At the end of the program running with the aid of outcome-based education, a student will be able to arrive at the following outcomes:

PO-1: Basic and Discipline specific knowledge: Apply knowledge of basic mathematics, science and engineering fundamentals and engineering specialization to solve the engineering problems.

PO-2: Problem analysis: Identify and analyze well-defined engineering problems using codified standard methods.

PO-3: Design/ development of solutions: Design solutions for well-defined technical problems and assist with the design of systems components or processes to meet specified needs.

PO-4: Engineering Tools, Experimentation and Testing: Apply modern engineering tools and appropriate technique to conduct standard tests and measurements.

PO-5: Engineering practices for society, sustainability and environment: Apply appropriate technology in context of society, sustainability, environment and ethical practices.

PO-6: Project Management: Use engineering management principles individually, as a team member or a leader to manage projects and effectively communicate about well-defined engineering activities.

PO-7: Life-long learning: Ability to analyze individual needs and engage in updating in the context of technological changes.

Course Outcomes

After completion of the course the students will be able to:

CO-1: Apply trigonometry and related basic concepts to solve applied technical problems.

CO-2: Demonstrate the ability to algebraically analyse basic functions used in Trigonometry.

CO-3: Use basic concepts of Differential Calculus to solve engineering related problems.

CO-4: Interpret the derivative of a function graphically, numerically and analytically.

CO-5: Demonstrate the ability to model real -life scenarios using functions.

CO-6: Communicate mathematical thinking coherently and clearly to students, peers, and others.

CO-7: Solve engineering related problems based on concepts of Algebra.

Course Outcome	Expected Mapping with Program Outcomes (1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation)						
	PO-1	PO-2	PO-3	PO-4	PO-5	PO-6	PO-7
CO-1	3	2	1	-	-	-	1
CO-2	3	2	-	-	-	-	1
CO-3	3	2	1	-	-	-	1
CO-4	2	2	-	-	-	-	1
CO-5	2	2	1	-	-	-	-
CO-6	2	2	1	-	-	-	2
CO-7	2	2	-	-	-	-	-

Abbreviations and Symbols

List of Abbreviations

Abbreviations	Full form
CO	Course Outcomes
cm	Centimeter
PO	Program Outcomes
t-Ratio	Trigonometrical ratio
UO	Unit Outcomes
UV	Ultra Violet
All STC	
All	All trigonometric t-ratios are positive
S	Sine and Cosec t-ratios are positive
T	Tan and Cot t-ratios are positive
C	Cos and Sec t-ratios are positive

List of Symbols

Symbols	Description
1^R	1 Radian
1^G	1 Grade
	1 Degree
1^0	1 Degree
$1^{'}$	1 Minute
$1^{'}$	1 Second
π	Pai
D_f	Domain of a function f
R_f	Range of a function f
$x \to a$	x approaches to a
$\bigcup$	Union
$\bigcap$	Intersection

Symbols	Description
$\subset$	Subset
$\in$	Belongs to
$\left[a,b\right]$	Closed interval
$\left(a,b\right)$	Open interval
$[a,b)$	Close – open interval
$(a,b]$	Open – close interval
ϕ	Empty Set
$[x]$	Greatest Integer
$\frac{d}{dx}(y)=\frac{dy}{dx}$	Derivative of w.r..t. x
i	Iota
$\bar{z}$	Conjugate of Z
$\lvert z\rvert$	Modulus of Z
$\mathrm{Re}\,l(z)$	Real part of Z
$\mathrm{Im}(z)$	Imaginary part of Z
$\arg(z)$	Argument of Z
$amp(z)$	Amplitude of Z
$cis\theta$	$\cos\theta+i\sin\theta$
$n!$	n factorial OR factorial n
${}^{n}P_{r}$	Number of permutations of n different objects taken r at a time
${}^{n}C_{r}$	Number of Combination of n different objects taken r at a time

List of Figures

Unit 1: Trigonometry

Contents

Fig. 1.1: Positive Angle 2
Fig. 1.2: Negative Angle 2
Fig. 1.3: Sign Convention 8
Fig. 1.4 Quadrant of Trigonometric Ratios of an Angle $\frac{A}{2}$ 18
Fig. 1.5: Graph of Sine Function 21
Fig. 1.6: Graph of Cosine Function 21
Fig. 1.7: Graph of Tangent Function 21
Fig. 1.8: Graph of Exponential 22
Fig. 1.9: Graph of y = sin x $\left(-90° \le x \le 90°\right)$ 22
Fig. 1.10: Graph of y= 3cos2 x $\left(-\frac{\pi}{4} \le x \le \frac{\pi}{4}\right)$ 23
Fig. 1.11: Graph of y = tan x 24
Fig. 1.12: Graph of $y = e^{2x}$ 25

Unit 2: Functions and Limit

Fig. 2.1: Function 39
Fig. 2.2: Not a Function 39
Fig. 2.3: Domain and Codomain 40
Fig. 2.4: Domain, Codomain and Range 40
Fig. 2.5: Graph of Constant Function 41
Fig. 2.6: Graph of Identity Function 42
Fig. 2.7: Graph of Modulus Function 42
Fig. 2.8: Graph of Greatest Integer Function 42
Fig. 2.9: Graph of Signum Function 43
Fig. 2.10: Graph of Reciprocal Function 43
Fig. 2.11: Graph of Logarithms 44
Fig. 2.12: Approach of x to a 45

Unit 4: Complex Numbers and Partial Fraction

Fig. 4.1: Conjugate Complex Number 97
Fig. 4.2: Modules of Complex Number 99
Fig. 4.3: Argument of Complex Number 101
Fig. 4.4: Principal Value of Argument 101
Fig. 4.5: Cartesian Representation of Complex Number 103

Unit 5: Permutation and Combination, Binomial Theorem

Fig. 5.1: Possibility Showing Multiplication-1 125
Fig. 5.2: Possibility Showing Multiplication-2 125
Fig. 5.3: Representation of Permutation 129
Fig. 5.4: Representation of Combination 131

List of Tables

Unit 1: Trigonometry

Table 1.1: Inter-conversions Degree-Grade-Radian 4

Table 1.2: t-ratios- for Allied angles 6

Table 1.3: t-ratios- for Various angles 6

Table 1.4: Sum(or difference) to Product formulae 13

Table 1.5: Short cut to remember Product to Sum formulae 15

Table 1.6: t-ratios of (A/2) formulae 17

Table 1.7: Values of sin x for $\left(-90° < x < 90°\right)$ 22

Table 1.8: Values of $y = 3 \cos 2x$ for $\left(-\pi/4 \le x \le \pi/4\right)$ 23

Table 1.9: Values of $y = \tan x$ for $\left(-60° \le x \le 60°\right)$ 23

Table 1.10: Values of $y = e^{2x}$ for $\left(-1° \le x \le 1\right)$ 24

Unit 2: Functions and Limit

Table 2.1: Events and their Scale of Impact 51

Unit 4: Complex Numbers and Partial Fraction

Table 4.1: Value of arguments of complex numbers 102

Table 4.2: Forms of rational functions and respective partial fraction 108

Guidelines for Teachers

To implement Outcome Based Education (OBE) knowledge level and skill set of the students should be enhanced. Teachers should take a major responsibility for the proper implementation of OBE. Teachers are suggested to use the special instructional strategies to accelerate the attainment of the various outcomes in this course. Some of the responsibilities (not limited to) for the teachers in OBE system may be as follows:

- Within reasonable constraint, they should manipulate time to the best advantage of all students.
- Massive open online courses (MOOCs) may be used to teach various topics/subtopics.
- Different types of teaching methods and media may be employed to develop the outcomes.
- About 10-15% of the topics/sub-topics which is relatively simpler or descriptive in nature is to be given to the students for self-directed learning and assess the development of the UOs/COs through classroom presentations.
- Assess the students only upon certain defined criterion without considering any other potential ineligibility to discriminate them.
- Guide student(s) in undertaking micro-projects.
- Ensure to grow the learning abilities of the students to a certain level before they leave the institute.
- Employ ICT Based Teaching Learning (Video Demonstration, Blog, Face book, Mobile learning)
- Teachers need to ensure to create opportunities and provisions for co-curricular activities.
- Encourage the students to develop their ultimate performance capabilities.
- Facilitate and encourage group work and team work to consolidate newer approach.
- Follow Blooms taxonomy in every part of the assessment.

Bloom's Taxonomy

Level	Teacher should Check	Student should be able to	Possible Mode of Assessment
Creating	Students ability to create	Design or Create	Mini project
Evaluating	Students ability to Justify	Argue or Defend	Assignment
Analysing	Students ability to distinguish	Differentiate or Distinguish	Project/Lab Methodology
Applying	Students ability to use information	Operate or Demonstrate	Technical Presentation/ Demonstration
Understanding	Students ability to explain the ideas	Explain or Classify	Presentation/Seminar
Remembering	Students ability to recall (or remember)	Define or Recall	Quiz

Guidelines for Students

Students should take equal responsibility for implementing the OBE. Other than the classroom learning, following are the suggested student-related co-curricular activities which can be undertaken to accelerate the attainment of the various outcomes in this course. Some of the responsibilities (not limited to) for the students in OBE system are as follows:

- Students should be well aware of each UO before the start of a unit in each and every course.
- Students should be well aware of each CO before the start of the course.
- Students should be well aware of each PO before the start of the program.
- Students should think critically and reasonably with proper reflection and action.
- Learning of students should base on real world problems relevant to content of the unit using free tutorials available on the internet.
- Learning of the students should be connected and integrated with practical and real-life consequences.
- Students should be well aware of their competency at every level of OBE.

Contents

Foreword iii
Acknowledgement v
Preface vii
Outcome Based Education ix
Course Outcomes x
Abbreviations and Symbols xi
List of Figures xiii
List of Tables xiv
Guidelines for Teachers xv
Guidelines for Students xvi

1. Trigonometry 1–36

Unit Specifics 1
Rationale 1
Pre-Requisites 2
Unit Outcomes 2
1.1 Introduction 2
1.1.1 System of Measurement of Angles 3
1.1.2 Units of Measurement of Angle 3
1.1.3 Relation between three Systems of Measurement of an Angle 3
1.2 Trigonometical Ratios of Allied Angle 6
1.2.1 t-ratios of $(-\theta)$ 7
1.2.2 t-ratios of $(90^o - \theta)$ 7
1.2.3 t-ratio of $(180^o - \theta)$ 7
1.3 Sum and Difference formula and their application 10
1.3.1 Sum and Difference Formulae 10
1.3.2 Formulae for the Trigonometric Ratios of Sum and Differences of two Angles 11
1.3.3 Applications of Sum-Difference Formulae 11
1.3.4 Applications of Product Formulae 13
1.4 t-ratios of Multiple Angles and Sub-Multiple Angles 17
1.4.1 t-ratios of 2A 17
1.4.2 t-ratios of 3A 17
1.4.3 t-ratios of (A/2) 17

1.5 Graph of the functions 20
Applications (Real Life / Industrial) 25
Unit Summary 26
Exercises 26
Know More 35
References and Suggested Readings 36

2. Functions and Limit 37–60

Unit Specifics 37
Rationale 37
Pre-Requisites 38
Unit Outcomes 38
2.1 Function 38
2.1.1 Definition of function 38
2.1.2 Domain, Co-Domain and Range of Function 40
2.1.3 Some Special Functions their Domain, Range and Graph 41
2.2 Limit of a function 44
2.2.1 Left and Right Limit 45
2.2.2 Existence of Limit 45
Applications (Real Life / Industrial) 51
Unit Summary 52
Exercises 53
Know More 58
References and Suggested Readings 59

3. Differential Calculus 61-90

Unit Specifics 61
Rationale 61
Pre-Requisites 62
Unit Outcomes 62
3.1 Derivative of function at a point 62
3.1.1 Derivative of Function 63
3.1.2 Differentiation of Some Standard Function by Definition 63
3.2 Algebra of Derivative of Functions 66
3.3 Differentiation of Composite Function (Chain Rule) 70
3.4 Differentiation of Trigonometric and Inverse Trigonometric Functions 72
3.5 Differentiation of Logarithmic and Exponential Functions 75

Applications (Real Life / Industrial) 77
Unit Summary 78
Exercises 79
Know More 88
References and Suggested Readings 89

4. Complex Numbers and Partial Fraction 91-122

Unit Specifics 91
Rationale 91
Pre-Requisites 92
Unit Outcomes 92
4.1 Definition and Algebra of Complex Number 93
4.1.1 Basic Concept of Complex Number 93
4.1.2 Real and Imaginary Parts of a Complex Number 94
4.1.3 Algebraic Operations with Complex Numbers 94
4.1.4 Properties of Algebraic Operations on Complex Numbers 94
4.1.5 Equality of Two Complex Numbers 96
4.2 Conjugate of a Complex Number 97
4.2.1 Conjugate Complex Number 97
4.2.2 Properties of Conjugate 97
4.3 Modulus of a Complex Number 99
4.3.1 Definition 99
4.3.2 Properties of Modulus 99
4.4 Argument (Amplitude) of a Complex Number 101
4.4.1 Definition 101
4.4.2 Principal value of arg (z) 101
4.5 Various representations of a Complex Number 103
4.5.1 Geometrical Representation (Cartesian Representation) 103
4.5.2 Trigonometrical (Polar) Representation 104
4.5.3 Conversion of One form to Another 104
4.6 De' Moivre's theorem 105
4.7 Partial Fractions 107
Applications (Real Life / Industrial) 111
Unit Summary 112
Exercises 112
Know More 120
References and Suggested Readings 121

5. Permutation and Combination, Binomial Theorem 123-154

Unit Specifics 123
Rationale 123
Pre-Requisite 124
Unit Outcomes 124
5.1 Fundamental Principle of Counting 124
5.1.1 Principle of Multiplication 124
5.1.2 Principle of Addition 126
5.2 Permutations 126
5.2.1 Permutations when all the Objects are Distinct 126
5.2.2 Factorial Notation 127
5.2.3 Permutation Under Various Case 128
5.3 Combinations 131
5.4 Binomial expression 133
5.4.1 Binomial Theorem for Positive Integral Index 133
5.4.2 Binomial Theorem for any Index 134
5.4.3 Problems on Approximation by the Binomial Theorem 135
Applications (Real Life / Industrial) 141
Unit Summary 142
Exercises 142
Know More 152
References and Suggested Readings 153

Appendices 155

References for Further Learning 156

CO and PO Attainment Table 157

Index 158

1 Trigonometry

UNIT SPECIFICS

This unit elaborately discusses the following topics:

- Concept of angles;
- Measurement of angles in degrees;
- Grades and radians and their conversions;
- t-ratios of Allied angles (without proof)

The applications-based problems are discussed for generating further curiosity and creativity as well as improving problem solving capacity.

Besides giving a large number of multiple-choice questions as well as questions of short and long answer types marked in two categories following lower and higher order of Bloom's taxonomy, assignments through a number of numerical problems, a list of references and suggested readings are given in the unit so that one can go through them for practice.

Based on the content, there is "Know More" section added. This section has been thoughtfully planned so that the supplementary information provided in this part becomes beneficial for the users of the book. This section mainly highlights further teaching and learning related to some interesting facts about Trigonometry, history of the development of the subject focusing the salient observations, Possible reasons to struggle teaching trigonometry to students, need to study Trigonometry, Field of education in which trigonometry is used, the simplest way to learn trigonometry. On the other hand, suggested Micro projects and brain storming questions create inquisitiveness and curiosity for the topics included in the unit.

RATIONALE

Trigonometry was originally developed to solve problems related to astronomy, but soon found applications to navigation and a wide range of other areas related to Science and Engineering. It is of great practical importance to builders, architects, surveyors and engineers and has many other applications. Civil and mechanical engineers use trigonometry to calculate torque and forces on objects. Engineers use trigonometry to decompose the forces into horizontal and vertical components that can be analyzed. Things like the generation of electrical current or a computer use angles in ways that are difficult to see directly, but that rely on the fundamental rules of trigonometry to work properly. Any time angles appear in a problem, the use of trigonometry usually will not be far behind.

PRE-REQUISITES

- Familiarity with Pythagoras' theorem.
- Basic knowledge of congruence and similarity of triangles.
- Knowledge of the basic properties of triangles, squares and rectangles.
- Facility with simple algebra and equations.
- Familiarity with the use of a calculator

UNIT OUTCOMES

List of outcomes of this unit are as follows:

U1-O1: Apply the concept of Sum and Factor formulae to solve the given problem(s)

U1-O2: Use the concept of Multiple and Sub- Multiple angle to solve related problem(s).

U1-O3: Solve equations involving given trigonometric ratios.

U1-O4: Make use of real-life scenarios to identify same kind of graph of trigonometric functions.

U1-O5: Interpret geometrically the given Graph of Trigonometric functions.

Unit Outcome	**Expected Mapping with Program Outcomes** *(1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation)*						
	CO-1	**CO-2**	**CO-3**	**CO-4**	**CO-5**	**CO-6**	**CO-7**
U1-O1	3	3	-	-	1	1	-
U1-O2	3	2	-	-	1	1	-
U1-O3	3	3	-	-	1	1	-
U1-O4	3	3	-	-	1	1	-
U1-O5	3	3	-	-	-	-	-

1.1 INTRODUCTION

Angle is a measure of rotation of a given ray about its initial point. The original ray is called the *initial side* and the final position of the ray after rotation is called the *terminal side* of the angle. The point of rotation is called the *vertex*. If the direction of rotation is anticlockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is *negative*

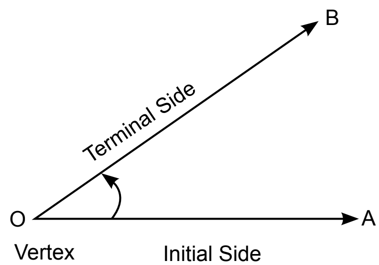

Fig. 1.1: Positive Angle

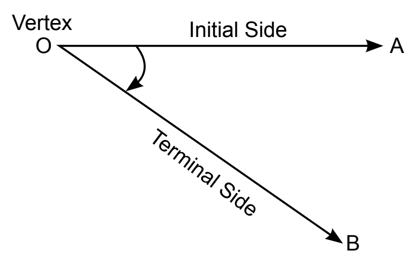

Fig. 1.2: Negative Angle

The word trigonometry is resulting from two greek words 'trigonon' and 'metron'. The word 'trigonon' means a triangle and the word 'metron' means a measure. Hence the word trigonometry means the study of properties of triangles. This includes the measurement of angles and lengths.

1.1.1 System of Measurement of Angles

There are two system for measuring angles

(1) Sexagesimal or English system: Therefore,

1 right angle = 90 degree $(=90^\circ)$

$1^\circ = 60$ minutes $(=60')$

$1' = 60$ seconds $(=60'')$

(2) Centesimal or French system: Therefore,

1 right angle = 100 grades $(=100^G)$

1 grade = 100 minutes $(=100')$

1 minute = 100 seconds $(=100'')$

1.1.2 Units of Measurement of Angle

The trigonometry is based on the measurement of angles.
There are three units of measurement for angles.

(1) Degree
(2) Grade and
(3) Radian

1.1.2.1 Degree

A protractor is the most common device used to measure angles. The simplest protractor comprises a semi-circular disk graduated in degree from 0° to 180°. The angle tool app is available only for Android users.

1.1.2.2 Grade

Grade is unit of measurement of an angle. Its defined as one hundredth of the right angle. i.e. There are 100 gradients in 90 degrees. In trigonometry the gradian also known as the "gon".

1.1.2.3 Radian

The angle subtended at the center of unit circle by a unit arc length on the circumference is called one radian. It is denoted by 1R.

1.1.3 Relation between three Systems of Measurement of an Angle

Let D be the number of degrees, R be the number of radians and G be the number of grades in an angle q, then

$$\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$$

Table 1.1: Inter-conversions Degree-Grade-Radian

$x^G = \left(\frac{9x}{10}\right)^D$	$x^D = \left(\frac{10x}{9}\right)^G$	$x^R = \left(\frac{180x}{p}\right)^D$
$x^G = \left(\frac{\pi x}{200}\right)^R$	$x^D = \left(\frac{\pi x}{180}\right)^R$	$x^R = \left(\frac{200x}{\pi}\right)^G$

*** Important points to be noticed:**
The value of π is approximately 22/7 OR 3.14.
This is the required relation between the three units of measurement of an angle.
Now, one radian = $\frac{180^o}{\pi}$ implies π radians = 180°
i.e., 1 radian = 57˚17’44.8” $\approx$ 57°17’45”

Example 1: Convert the following in remaining two units of measurement of an angle.

(i) 30^o (ii) 2^G (iii) $\left(\frac{\pi}{3}\right)^R$

Solution:

(i) we know that $x^D = \left(\frac{10x}{9}\right)^G$

$$\therefore 30^0 = \left(\frac{10 \times 30}{9}\right)^G = \left(\frac{300}{9}\right)^G$$

$$\text{and } 30^0 = \left(\frac{\pi \times 30}{180}\right)^R = \left(\frac{\pi}{6}\right)^R$$

(ii) We know that, $x^G = \left(\frac{9x}{10}\right)^D$

$$\therefore 2^G = \left(\frac{9 \times 2}{10}\right)^D = (1.8)^D = 1°48'$$

$$\text{and } x^G = \left(\frac{\pi x}{200}\right)^R$$

$$\therefore 2^G = \left(\frac{\pi \times 2}{200}\right)^R = \left(\frac{\pi}{100}\right)^R$$

(iii) We know that, $x^R = \left(\frac{200x}{\pi}\right)^G$

$$\therefore \left(\frac{\pi}{3}\right)^R = \left(\frac{200\times\frac{\pi}{3}}{\pi}\right)^G = \left(\frac{200}{3}\right)^G$$

$$\therefore\ x^R = \left(\frac{180x}{\pi}\right)^D$$

$$\therefore \left(\frac{\pi}{3}\right)^R = \left(\frac{180\times\frac{\pi}{3}}{\pi}\right)^D = 60^\circ$$

Example 2: Convert $40^\circ 20'$ into radian measure

Solution: We know that $180^\circ = \pi$ radian.

Hence,

$$40^\circ 20' = 40\frac{1}{3}\,degree = \frac{\pi}{180}\times\frac{121}{3}\,radian = \frac{121\pi}{540}\,radian$$

Example 3: Convert 6 radian into Degree measure

Solution: We know that $\pi\,\text{radian} = 180^\circ$

Hence

$$6\,radian = \frac{180}{\pi}\times 6 = \frac{1080\times 7}{22}\,degree = 343\frac{7}{11}\,degree = 343 + 7\times\frac{60}{11}\,minutes.\ [\text{as } 1^\circ = 60'\ ]$$

$$= 343^\circ + 38' + \frac{2}{11}\,min$$

$$= 343^\circ + 38' + \frac{2}{11}\times 60\,second = 343^\circ 38' 11''\ \text{approximately}$$

Example 4: Which of the following relations is correct?

(a) $\sin 1 < \sin 1^\circ$ (b) $\sin 1 > \sin 1^\circ$ (c) $\sin 1 = \sin 1^\circ$ (d) $\frac{\pi}{180} sin1 = sin1^o$

Solution:

(b) The true relation is $\sin 1 > \sin 1^\circ as\, 1 Radian = 57^\circ\, approximately$

Since value of $\sin\theta$ is increasing $\left[0 \to \frac{\pi}{2}\right]$.

1.2 TRIGONOMETICAL RATIOS OF ALLIED ANGLE

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90°

Table 1.2: T-ratios- for Allied angles

Allied Angles / Trignometric Ratios	sin θ	cos θ	tan θ
(–θ)	–sin θ	cos θ	–tan θ
$(90°-\theta)$ or $\left(\frac{\pi}{2}-\theta\right)$	cos θ	sin θ	cot θ
$(90°+\theta)$ or $\left(\frac{\pi}{2}+\theta\right)$	cos θ	–sin θ	–cot θ
$(180°-\theta)$ or $(\pi-\theta)$	sin θ	–cos θ	–tan θ
$(180°+\theta)$ or $(\pi+\theta)$	–sin θ	–cos θ	tan θ
$(270°-\theta)$ or $\left(\frac{3\pi}{2}-\theta\right)$	–cos θ	–sin θ	cot θ
$(270°+\theta)$ or $\left(\frac{3\pi}{2}+\theta\right)$	–cos θ	sin θ	–cot θ
$(360°-\theta)$ or $(2\pi-\theta)$	–sin θ	cos θ	–tan θ
$(360°+\theta)$ or $(2\pi+\theta)$	sin θ	cos θ	tan θ

Trignometric rations for various angles

Table 1.3: T-ratios- for Various angles

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	
sin θ	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	–1	0
cos θ	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	–1	0	1

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	
tan θ	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	0	∞	0

1.2.1 t-ratios of (–θ)

sin (–θ) = –sinθ

cos(-θ) = cosθ

tan(–θ) = –tanθ

cosec (–θ) = –cosecθ

sec (–θ) = secθ

cot (–θ) = –cotθ

1.2.2 t-ratios of (90° – θ)

sin (90° – θ) = cosθ

cos (90° – θ) = sinθ

tan (90° – θ) = cotθ

cot (90° – θ) = tanθ

sec (90° – θ) = cosecθ

cosec (90° – θ) = secθ

1.2.3 t-ratio of (180° – θ)

sin (180° – θ) = sinθ,

cos (180° – θ) = –cosθ

tan (180° – θ) = –tanθ,

cot (180° – θ) = – cotθ

sec (180° – θ) = –secθ,

cosec (180° – θ) = cosecθ

In general, for t-ratios of allied angles use the following rules:

Rule:1 If the angle θ (θ is an acute angle) is increased or decreased by a multiple of 360° (i.e. $2\pi R$) than t-ratio of an angle remain same.

Rule: 2 t-ratios of an angle –θ, π ± θ, 2π ± θ, ... remain same.

Rule :3 t-ratios of an angle $\frac{\pi}{2} \pm \theta, \frac{3\pi}{2} \pm \theta, \frac{5\pi}{2} \pm \theta \ldots$ changes as given below.

sin ↔ cos (sin changes to cos and vice versa)

tan ↔ cot

sec ↔ cosec

Rule: 4 For fixing a sign on RHS use AllSTC.

"All students Take Care"(**AllSTC**) see the figure given below.

All = all t-ratios are positive

S – Only sin and cosec are positive

T – Only tan and cot are positive

C – Only cos and sec are positive

Key - 1 Find the quadrant in which the given angle is

Key - 2 Identify the sign (positive or negative) of the given t-ratios using Rule 4.

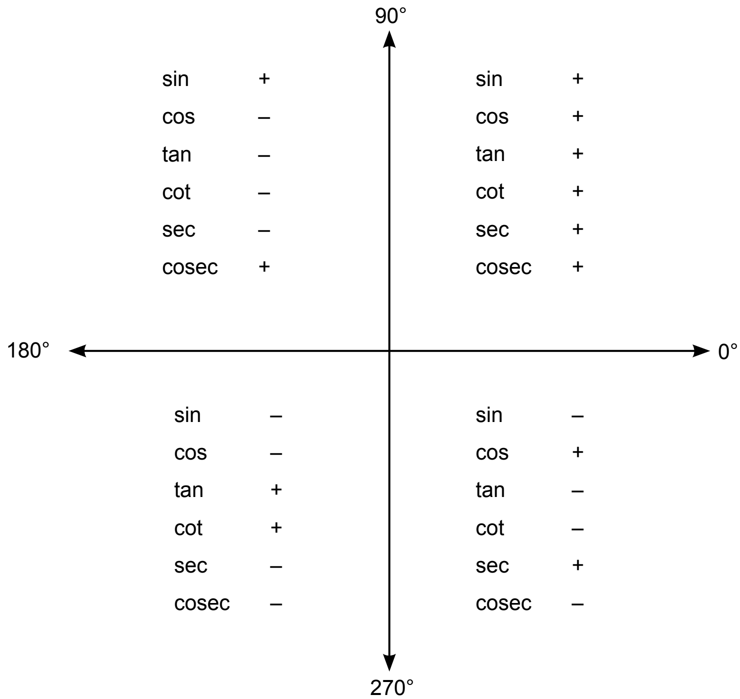

Fig. 1.3: Sign Convention

Example 5: Find the values of the following t-ratios

(i) $\sin 480^o$ (ii) $\tan(\pi + \theta)$ (iii) $sec\left(\dfrac{3\pi}{2} - \theta\right)$

Solution:

(i) $\sin(480^o) = \sin(450^o + 30^o)$

$= \cos 30^o = \dfrac{\sqrt{3}}{2}$ {2^{nd} Quadrant and $\left(\dfrac{\pi}{2} + \theta\right)$

(ii) $\tan(\pi + \theta) = \tan\theta$ {3^{rd} Quadrant and $\pi + \theta$ form

(iii) $sec\left(\dfrac{3\pi}{2} - \theta\right) = -\operatorname{cosec}\theta$ {3^{rd} Quadrant and $\left(\dfrac{\pi}{2} - \theta\right)$ form

Example 6: Find the value of following t-ratio

(i) $cosec^2\left(\dfrac{7\pi}{6}\right)$ (ii) $cot\left(\dfrac{5\pi}{6}\right)$

Solution:

(i) $cosec^2\left(\dfrac{7\pi}{6}\right) = \left[cosec\dfrac{7\pi}{6}\right]^2$

$$=\left[cosec\left(\pi+\frac{\pi}{6}\right)\right]^2=\left[-cosec\left(\frac{\pi}{6}\right)\right]^2$$

$$=\left[-\frac{1}{2}\right]^2=\frac{1}{4}$$

(ii) $$cot\left(\frac{5\pi}{6}\right)=cot\left(\pi-\frac{\pi}{6}\right)$$

$$=-cot\frac{\pi}{6}=-\sqrt{3}$$

Example 7: Prove that $2sin^2\frac{3\pi}{4}+2cos^2\frac{\pi}{4}+sec^2\frac{\pi}{3}=6$

Solution:

L.H.S = $2sin^2\frac{3\pi}{4}+2cos^2\frac{\pi}{4}+sec^2\frac{\pi}{3}$

$$=2\left(sin\frac{3\pi}{4}\right)^2+2\left(cos\frac{\pi}{4}\right)^2+\left(sec\frac{\pi}{3}\right)^2$$

$$=2\left(sin\frac{\pi}{4}\right)^2+2\left(cos\frac{\pi}{4}\right)^2+\left(sec\frac{\pi}{3}\right)^2$$

$$\therefore sin\frac{3\pi}{4}=sin\left(\pi-\frac{\pi}{4}\right)=sin\frac{\pi}{4}$$

$$=2\left(\frac{1}{\sqrt{2}}\right)^2+2\left(\frac{1}{\sqrt{2}}\right)^2+(2)^2$$

= 1 + 1 + 4 = 6

= R.H.S.

Example 8: Prove that tan 225° cot 405° + tan 1485° cot 315° = 0

Solution:

tan 225° = tan (180° + 45°)

= tan 45° {3^{rd} Quadrant and π + θ form

= 1

Similarly, cot 405° = cot (360° + 45°) = cot 45°= 1

tan 1485° = tan (1440° + 45°) {1440° = 4 x 360°)

= tan 45° = 1

cot 315° = cot (270° + 45°)

= – tan 45°

= –1

L.H.S. = tan225° cot405° + tan 1485° cot 315°

= (1) (1) + (1) (−1) = 1 − 1= 0 = R.H.S.

Example 9: Prove that

$$\frac{\sin\left(\theta-\frac{\pi}{2}\right)}{\cos(\pi-\theta)}+\frac{\tan\left(\frac{\pi}{2}-\theta\right)}{\cot(\pi+\theta)}+\frac{\operatorname{cosec}\left(\frac{\pi}{2}+\theta\right)}{\sec(2\pi-\theta)}=3$$

Solution :

$$\sin\left(\theta-\frac{\pi}{2}\right)=sin\left[-\left(\frac{\pi}{2}-\theta\right)\right]=-\sin\left(\frac{\pi}{2}-\theta\right) \quad \{\therefore \sin(-\theta=sin\,\theta$$

$$=-cos\theta$$

$$\cos(\pi-\theta)=-cos\theta,$$

$$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta,$$

$$\cot(\pi+\theta)=cot\theta,$$

$$\operatorname{cosec}\left(\frac{\pi}{2}+\theta\right)=sec\theta,$$

$$\sec(2\pi-\theta)=sec\theta$$

$$\text{Now, L.H.S. }=\frac{\sin\left(\theta-\frac{\pi}{2}\right)}{\cos(\pi-\theta)}+\frac{\tan\left(\frac{\pi}{2}-\theta\right)}{\cot(\pi+\theta)}+\frac{\operatorname{cosec}\left(\frac{\pi}{2}+\theta\right)}{\sec(2\pi-\theta)}$$

$$=\frac{-\cos(\theta)}{-\cos(\theta)}+\frac{\cot\theta}{\cot\theta}+\frac{sec\theta}{sec\theta}=1+1+1=3 \text{ = R.H.S.}$$

Example 10: For the Δ PQR, prove that sin (Q + R) = sin P

Solution:

We know that for Δ PQR

m ∠ P + m ∠ Q + m ∠ R = 180° = π

∴ L.H.S. = sin (Q + R)

= sin (π − P)

= sin P = R. H. S.

1.3 SUM AND DIFFERENCE FORMULA AND THEIR APPLICATION

1.3.1 Sum and Difference Formulae

A sum formula is a type of formula which helps to simplify a t-ratios of the sum of the angles. One of the main obstacles to cracking a problem in trigonometry is to transform the problem into a simpler form which is easy to solve. Sum and Difference formulas play a vital role to resolve this situation.

1.3.2 Formulae for the Trigonometric Ratios of Sum and Differences of two Angles

(1) $\sin(A+B)=\sin A\cos B+\cos A\sin B$

(2) $\sin(A-B)=\sin A\cos B-\cos A\sin B$

(3) $\cos(A+B)=\cos A\cos B-\sin A\sin B$

(4) $\cos(A-B)=\cos A\cos B+\sin A\sin B$

(5) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

(6) $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

(7) $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot A+\cot B}$

(8) $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$

(9) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A$

(10) $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A$

1.3.3 Applications of Sum-Difference Formulae

Example 11: Evaluate *sin* 75°

Solution:

We know that $sin\left(A+B\right)=sinA.cosB-cosA.sinB$

$$\therefore sin75^\circ=sin\left(45^\circ+30^\circ\right)$$

$$=\sin 45^\circ.\cos 30^\circ+\cos 45^\circ.\sin 30^\circ$$

$$=\frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}.\frac{1}{2}$$

$$=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}+1}{2}\right)$$

$$=\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\left(\frac{\sqrt{3}+1}{2}\right)$$

$$=\frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}+1}{2}\right)$$

$$=\frac{\sqrt{6}+\sqrt{2}}{4}$$ {Rationalized

Example 12: Evaluate tan 15°

Solution:

$$\tan(A - B) = \frac{\tan A - tanB}{1 + tanAtanB}$$

$\therefore$ tan 15° = tan (45° – 30°)

$$= \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \cdot \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} - 1}{\sqrt{3} - 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

Example 13: Prove that tan 57° = $\dfrac{cos12° + sin12°}{cos12° - sin12°}$

Solution:

L.H.S. = tan 57° = tan (45° + 12°)

$$= \frac{\tan 45° + \tan 12°}{1 - \tan 45° . \tan 12°}$$

$$= \frac{1 + \tan 12°}{1 - \tan 12°}$$

$$= \frac{1 + \frac{\sin 12°}{\cos 12°}}{1 - \frac{\sin 12°}{\cos 12°}}$$

$$= \frac{\cos 12° + \sin 12°}{\cos 12° - \sin 12°} = \text{R.H.S.}$$

Example 14: Prove that tan 50° = tan 40° + 2 tan 10°

Solution:

tan 50° = tan (40° + 10°)

$\therefore$ tan 50° – tan 50° tan 40° tan 10° = tan 40° + tan 10°

$\therefore$ tan 50° – cot 40° tan 40° tan 10° = tan 40° + tan 10°

$$\left\{ \because \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \text{ and } \tan\theta \cdot \cot\theta = 1 \right.$$

$\therefore$ tan 50° – tan 10° = tan 40° – tan 10° ($\therefore$ tan θ + cot θ = 1

$\therefore$ tan 50° = tan 40° + 2tan 10°

Hence proved.

Example 15: Evaluate $cos^2\left(\frac{\pi}{4}+x\right)-sin^2\left(\frac{\pi}{4}-x\right)$

Solution:

$$cos^2 A - sin^2 B = cos(A+B).cos(A-B)$$

$$\therefore cos^2\left(\frac{\pi}{4}+x\right)-sin^2\left(\frac{\pi}{4}-x\right) = cos\left[\frac{\pi}{4}+x+\frac{\pi}{4}-x\right].cos\left[\frac{\pi}{4}+x-\frac{\pi}{4}+x\right]$$

$$= cos\left[\frac{2\pi}{4}\right].\cos[2x]$$

$$= 0.cos2x \; = 0$$

1.3.4 Applications of Product Formulae

Sum to product trigonometric formulas can be very supportive in abridging a trigonometric expression by taking the sum and converting it into a product.

1.3.4.1 Formulae to Transform the Sum or Difference into Product

Therefore, we find out the formulae to transform the sum or difference into product.

Table 1.4: Sum (or difference) to Product formulae

$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$	$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$
$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$	$\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}p$ or $\cos C - \cos D = 2\sin\frac{D+C}{2}\sin\frac{D-C}{2}$

Example 16: $\sin 65° + \cos 65° = \sqrt{2}\cos 20°$

Solution:

$$\text{LHS} = \sin 65° + \cos 65°$$

$$= \sin 65° + \cos 25° \quad [\therefore \cos 65° = \cos(90° - 25°) = \sin 25°]$$

$$= 2sin\left(\frac{65°+25°}{2}\right).cos\left(\frac{65°-25°}{2}\right)$$

$$= 2\sin 45°.cos20°$$

$$= 2\frac{1}{\sqrt{2}.}\cos 20°$$

$$= \sqrt{2}\cos 20° = \text{RHS}$$

Example 17: Prove that $\cot 2\theta + tan\theta = cosec\, 2\theta$

Solution:

$$\text{LHS} = \cot 2\theta + tan\theta = \frac{\cos 2\theta}{\sin 2\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos 2\theta.\cos\theta + \sin 2\theta.sin\theta}{\sin 2\theta.cos\theta}$$

$$= \frac{cos(2\theta - \theta)}{\sin 2\theta.cos\theta} \quad [\therefore \cos A.\cos B + \sin A.\sin B = \cos(A - B)]$$

$$= \frac{\cos\theta}{\sin 2\theta.\cos\theta}$$

$$= \frac{1}{\sin 2\theta}$$

$$= cosec 2\theta$$

$$= \text{R.H.S.}$$

Example 18: Prove that $\frac{sin(A-B)}{cosA.cosB} + \frac{sin(B-C)}{cos\,Bcos\,C} + \frac{sin(C-A)}{cos\,C\,cos\,A} = 0$

Solution:

First we simplify first term of LHS

$$\frac{sin(A-B)}{cosA.cosB} = \frac{sin\,Acos\,B - cos\,A\,sinB}{cos\,A\,cos\,B}$$

$$= \frac{sin\,A\,cos\,B}{cos\,A\,cos\,B} - \frac{cos\,A\,sin\,B}{cos\,A\,cos\,B}$$

$$= tan\,A - tan\,B$$

Similarly, $\frac{sin(B-C)}{cos\,Bcos\,C} = tan\,B - tan\,A$

and $\frac{sin(C-A)}{cos\,C\,cos\,A} = tanC - tan\,A$

Now, LHS = $\frac{sin(A-B)}{cosA.cosB} + \frac{sin(B-C)}{cos\,Bcos\,C} + \frac{sin(C-A)}{cos\,C\,cos\,A}$

$$= tan\,A - tan\,B + tan\,B - tan\,A + tanC - tan\,A$$

$$= 0$$

= RHS

Example 19: Evaluate $sin\frac{7\pi}{12}.cos\frac{\pi}{4} - cos\frac{7\pi}{12}.sin\frac{\pi}{4}$

Solution: $sin\frac{7\pi}{12}.cos\frac{\pi}{4} - cos\frac{7\pi}{12}.sin\frac{\pi}{4} = sin\left(\frac{7\pi}{12} - \frac{\pi}{4}\right)$

$\left[\sin A\cos B - \cos B.\sin A = \sin(A-B)\right]$

$$= sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Example 20: If $3\cot A\cot B = 1$, prove that $\cos(A-B) + 2\cos(A+B) = 0$

Solution:

Given that,

$3\cot A\cot B = 1$,

$\frac{3cos\,A.\cos B}{\sin A.\sin B} = 1$

$= \frac{cos\,A.\cos B}{\sin A.\sin B} = \frac{1}{3}$

$\therefore \frac{cos\,A.\cos B + \sin A.\sin B}{\cos A.\cos B - \sin A.\sin B} = \frac{1+3}{1-3}$

$= \frac{\cos(A-B)}{\cos(A+B)} = \frac{4}{-2} = -2$

$\therefore \cos(A-B) = -2\cos(A+B)$

$\therefore \cos(A-B) + 2\cos(A+B) = 0$

Hence proved.

1.3.4.2 Formulae to Transform the Product into Sum or Difference

Product to sum or difference trigonometric formulas can be very helpful in shortening a trigonometric expression by taking the product and converting it into a sum or difference.

- $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$
- $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$
- $2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$
- $-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$ OR

  $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

Short cut key to remember above formulae

Table 1.5: Short cut to remember Product to Sum formulae

2SC = S + S	2CC = C + C
2CS = S – S	–2SS = C – C

Example 21: Prove that $\sin 10° \sin 30° \sin 50° \sin 70° = \frac{1}{16}$

Solution:

$$\text{L.H.S.} = \sin 10° \sin 30° \sin 50° \sin 70°$$

$$= \frac{1}{2}\left(\sin 10° \sin 50° \sin 70°\right)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \left(2\sin 50° \sin 10°\right) \times \sin 70°$$

$$= \frac{1}{4}\left[cos\left(50° - 10°\right) - cos\left(50° + 10°\right)\right] \times \sin 70°$$

$$\left[2\sin\alpha\cos\beta = \cos\left(\alpha - \beta\right) - \cos\left(\alpha + \beta\right)\right]$$

$$= \frac{1}{4} \times \left[\cos 40° - \cos 60°\right] \times \sin 70°$$

$$= \frac{1}{4} \times \left[\cos 40° - \frac{1}{2}\right] \times \sin 70°$$

$$= \frac{1}{4} \times \left[\cos 40° \sin 70° - \frac{1}{2}\sin 70°\right]$$

$$= \frac{1}{4} \times \left[\frac{1}{2}\left(2\sin 70°.\cos 40°\right) - \frac{1}{2}\sin 70°\right]$$

$$= \frac{1}{4}\left\{\frac{1}{2}\left[\sin\left(70° + 40°\right) + \sin\left(70° - 40°\right)\right] - \frac{1}{2}\sin 70°\right\}$$

$$= \frac{1}{8}\left\{\sin 110° + \sin 30° - \sin 70°\right\}$$

$$= \frac{1}{8}\left\{\sin 70° + \frac{1}{2} - \sin 70°\right\} \qquad \left\{\because \sin 110° = \sin\left(180° - 70°\right) = \sin 70°\right\}$$

$$= \frac{1}{16} = RHS$$

Example 22: Prove that $2cos\frac{5\pi}{12}.cos\frac{\pi}{12} = \frac{1}{2}$

Solution:

$$\text{L.H.S.} = 2cos\frac{5\pi}{12}.cos\frac{\pi}{12}$$

$$= 2cos\left(\frac{5\pi}{12} + \frac{\pi}{12}\right) + cos\left(\frac{5\pi}{12} - \frac{\pi}{12}\right)$$

$$\left[\because 2CC = C + C = cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = 0 + \frac{1}{2} = \frac{1}{2}\right]$$

= R.H.S.

1.4 t-RATIOS OF MULTIPLE ANGLES AND SUB-MULTIPLE ANGLES $\left(2A, 3A, \frac{A}{2}\right)$

We know that when there is a single angle then the computation is easy rather than multiple and submultiple angles. There are various formulae (Identities?) for multiple and sub-multiple angles. These identities are useful into solution of intricate trigonometric equations. These are also useful to find the t-ratios of negative angles. In this section we illustrate t-ratios of multiple and sub-multiple angles.

1.4.1 t-ratios of 2A

$$sin2A = 2sinA.cosA = \frac{2tan\,A}{1 + tan^2 A}$$

$$\cos 2A = cos^2 A - sin^2 A = 2cos^2 A - 1$$

$$= 1 - 2sin^2 A = \frac{2tan\,A}{1 + tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}, where A \neq (2n+1)\frac{\pi}{4}$$

1.4.2 t-ratios of 3A

$$sin3A = 3sin\,A - 4sin^3 A$$

$$cos3A = 4cos^3 A - 3cos\,A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}, \text{ where } A \neq (2n+1)\frac{\pi}{6}$$

1.4.3 t-ratios of (A/2)

Table 1.6: T-ratios of (A/2) formulae

$sinA = 2sin\frac{A}{2}cos\frac{A}{2} = \frac{2tan\frac{A}{2}}{1 + tan^2\frac{A}{2}}$
$cosA = cos^2\frac{A}{2} - sin^2\frac{A}{2} = 2cos^2\frac{A}{2} - 1 = 1 - 2sin^2\frac{A}{2} = \frac{1 - tan^2\frac{A}{2}}{1 + tan^2\frac{A}{2}}$

$tan\frac{A}{2}=\sqrt{\frac{1-cosA}{1+cosA}}$
$tanA=\frac{2tan\frac{A}{2}}{1-tan^2\frac{A}{2}}$
$\tan\frac{A}{2}=\frac{\pm\sqrt{\tan^2 A+1}-1}{\tan A}=\tan\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{1+\cos A}}$
$\cot\frac{A}{2}=\pm\sqrt{\frac{1+\cos A}{1-\cos A}}$

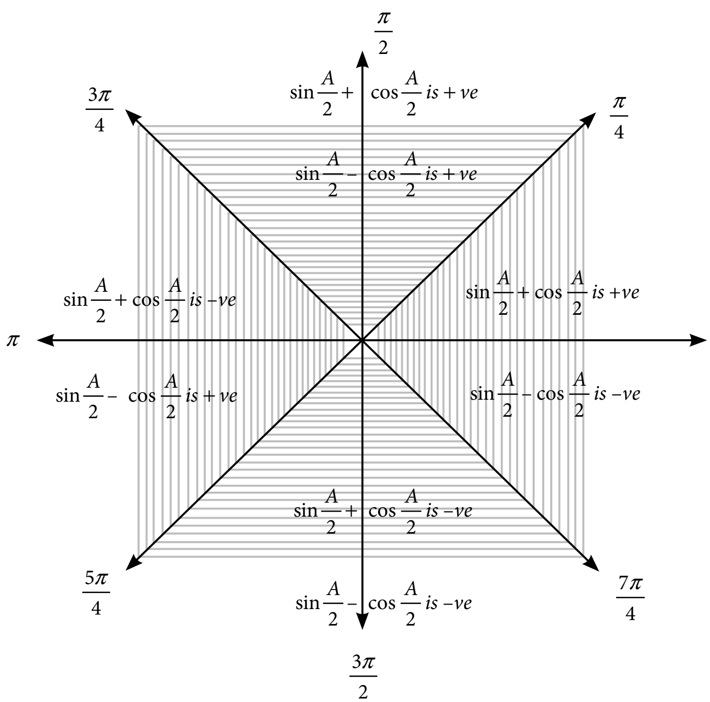

Fig. 1.4: Quadrant of Trigonometric Ratios of an Angle $\frac{A}{2}$

Example 23: Prove that $\frac{sin2\theta}{1+cos2\theta}=tan\theta$

Solution:

$$\text{L.H.S. } \frac{sin2\theta}{1+cos2\theta}=\frac{2sin\theta cos\theta}{1+2cos^2\theta-1}$$

$$= \frac{2sin\,\theta cos\theta}{2cos^2\theta}$$

$$= \frac{sin\,\theta}{cos\,\theta} = tan\theta$$

= R.H.S

Example 24: Prove that $\frac{1+\sin 2\theta}{1-\sin 2\theta} = tan^2\left(\frac{\pi}{4}+\theta\right)$

Solution:

$$\text{L.H.S.} = \frac{1+\sin 2\theta}{1-\sin 2\theta} = \frac{1+\frac{2tan\theta}{1+tan^2\theta}}{1-\frac{2tan\theta}{1+tan^2\theta}}$$

$$= \frac{\frac{1+tan^2\theta+2tan\theta}{1+tan^2\theta}}{\frac{1+tan^2\theta-2tan\theta}{1+tan^2\theta}} = \frac{1+tan^2\theta+2tan\theta}{1+tan^2\theta-2tan\theta} =$$

$$= \frac{(1+\tan\theta)^2}{(1-\tan\theta)^2} = \left(\frac{1+\tan\theta}{1-\tan\theta}\right)^2$$

$$= \left[tan\left(\frac{\pi}{4}+\theta\right)\right]^2 = tan^2\left(\frac{\pi}{4}+\theta\right)$$

= R.H.S.

Example 25: Prove that $\frac{1+\sin 2A-\cos 2A}{1+\sin 2A+\cos 2A} = \tan A$

Solution:

$$\text{L.H.S.} = \frac{1+\sin 2A-\cos 2A}{1+\sin 2A+\cos 2A}$$

$$= \frac{1+2\sin A cos\,A-\left(1-2sin^2 A\right)}{1+2\sin A cosA+\left(2cos^2 A-1\right)}$$

$$= \frac{1+2\sin A cos\,A-1+2sin^2 A}{1+2\sin A cosA+2cos^2 A-1}$$

$$= \frac{2\sin A cos\,A+2sin^2 A}{2\sin A cosA+2cos^2 A}$$

$$= \frac{2\sin A\left(cos\,A + sinA\right)}{2\cos A\left(cos\,A + sinA\right)}$$

$$= \frac{\sin A}{\cos A} = \tan A$$

$= \text{R.H.S}$

Example 26: Prove that $\dfrac{\sin 3\theta - \sin\theta}{\cos 3\theta + \cos\theta} = \tan\theta$

Solution:

$$\text{L.H.S.} = \frac{\sin 3\theta - \sin\theta}{\cos 3\theta + \cos\theta}$$

$$= \frac{3\sin\theta - 4\sin^3\theta - \sin\theta}{4\cos^3\theta - 3\cos\theta + \cos\theta}$$

$$= \frac{2\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 2\cos\theta}$$

$$= \frac{2\sin\theta(1 - 4\sin^2\theta)}{2\cos\theta(2\cos^2\theta - 1)}$$

$$= \frac{2\sin\theta cos2\theta}{2cos\theta cos2\theta} = \frac{\sin\theta}{\cos\theta} = tan\theta$$

$= \text{R.H.S}$

Example 27: Prove that $\cos 6A = 32cos^6 A - 48\,cos^4 A + 18\,cos^6 A - 1$

Solution:

$$\text{L.H.S.} = \cos 6A$$

$$= 2cos^2 3A - 1 \qquad \left[\because \cos 2A = 2cos^2 A - 1\right]$$

$$= 2\left(4cos^3 A - 3cosA\right)^2 - 1$$

$$= 2\left(16cos^6 A - 24cos^4 A + 9cos^2 A\right) - 13$$

$$= 32cos^6 A - 48cos^4 A + 18cos^2 A - 1$$

$= \text{R.H.S.}$

1.5 GRAPH OF THE FUNCTIONS

$sinx, \cos x, \tan x, e^x$

The graph of a function is often a useful way of visualizing the relationship of the function models, and manipulating a mathematical expression for a function which can throw light on the function's properties. Functions presented as expressions can model many important phenomena.

Graphs are used for modelling many different natural and mechanical phenomena (populations, waves, engines, acoustics, electronics, UV intensity, growth of plants, etc). The trigonometric graphs in this chapter are periodic, which means the shape repeats itself exactly after a certain amount of time.

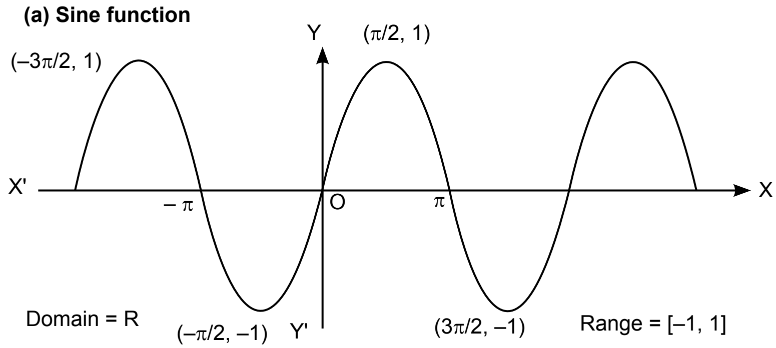

Fig. 1.5: Graph of Sine Function

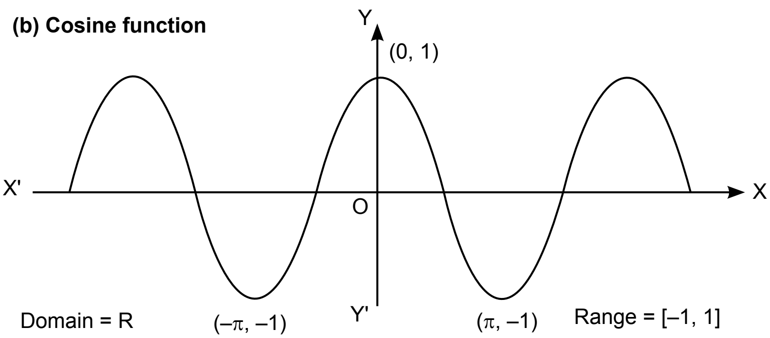

Fig. 1.6: Graph of Cosine Function

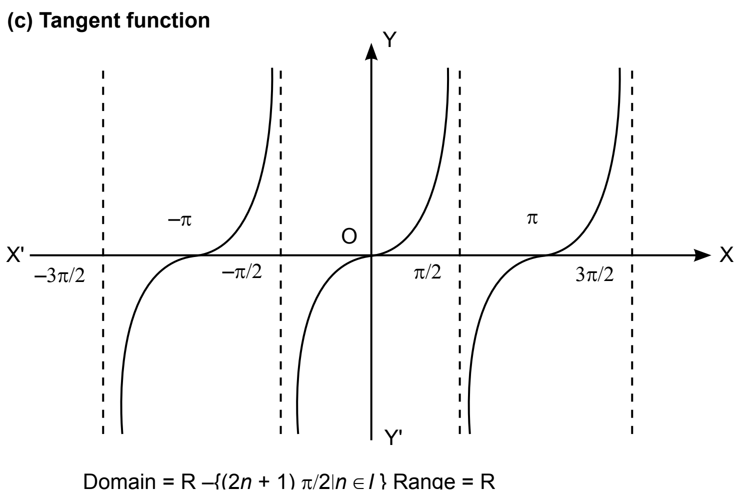

Fig. 1.7: Graph of Tangent Function

(d) raph of Exponential function: $y = e^x$

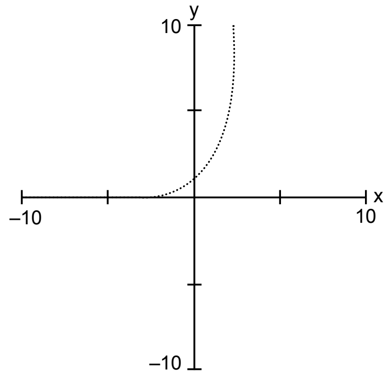

Fig. 1.8: Graph of Exponential

Example 28: Draw the graph of $y = sinx\left(-90° < x < 90°\right)$

Solution:

Here $y = sinx$ and we have to choose the values of x from $-90°\, to\, 90°$

Let, $x = -60°$ then $y = \sin(-60°) = -\sin(60°) = -0.86$

Similarly find the values of y with respect to the values of x from the given interval.

Now prepare a table as bellow and plot the points on to the graph paper. Now join all the points by rough hand.

It's a required graph.

Table 1.7: Values of sin x for $\left(-90° < x < 90°\right)$

X	–60°	–30°	0°	30°	60°
sinx	-0.86	-0.5	0	0.5	0.86

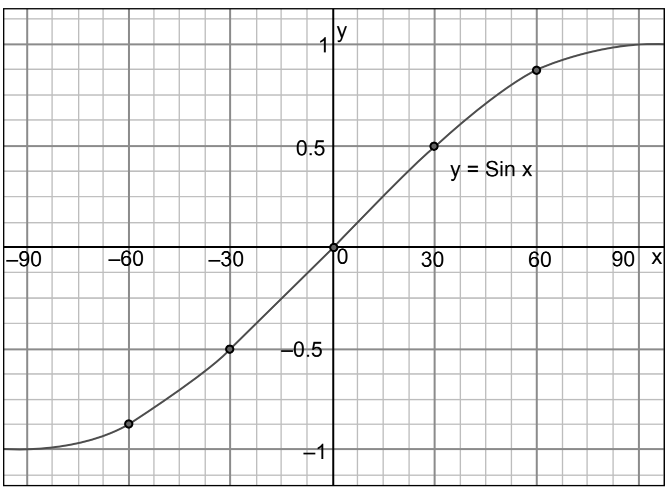

Fig.1.9: Graph of y = sin x $\left(-90° \le x \le 90°\right)$

Example 29: Draw the graph of $y = 3\cos 2x$ $\left(-\pi/4 \le x \le \pi/4\right)$

Solution:

Table 1.8: Values of y = 3 cos2*x* for $\left(-\pi/4 \le x \le \pi/4\right)$

X	–45°	–30°	0°	30°	45°
Y = 3cos2x	0	-1.5	3	1.5	0

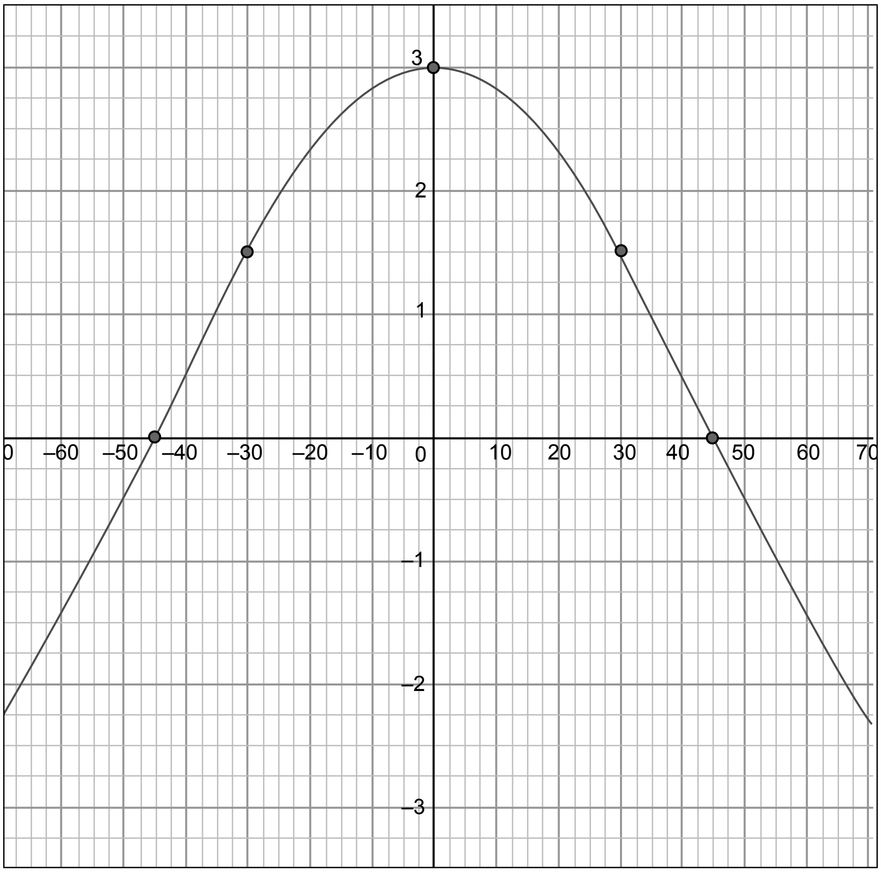

Fig. 1.10: Graph of y= 3cos2 x $\left(-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}\right)$

Example 30: Draw the graph of $y = tanx$

Solution:

Table 1.9: Values of $y = \tan x$ for $\left(-60° \le x \le 60°\right)$

X	–60°	–30°	0°	30°	60°
Tanx	–1.73	–0.58	0	0.58	1.73

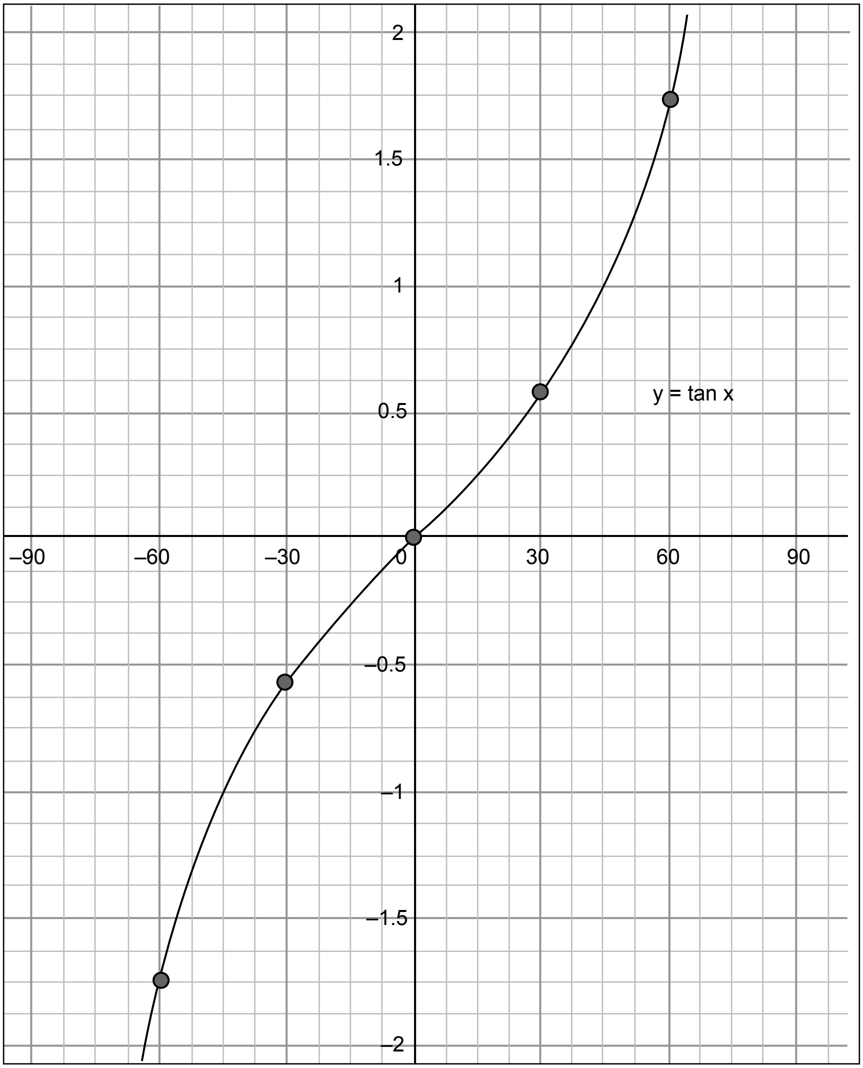

Fig. 1.11: Graph of $y = tan^{x}$

Example 31: Draw the graph of $y = e^{2x}$

Solution:

Table 1.10: Values of $y = e^{2x}$ for $\left(-1^{\circ} \leq x \leq 1\right)$

X	**–1**	**–1/2**	**0**	**1/2**	**1**
e^{2x}	0.1	0.4	1	2.8	7.4

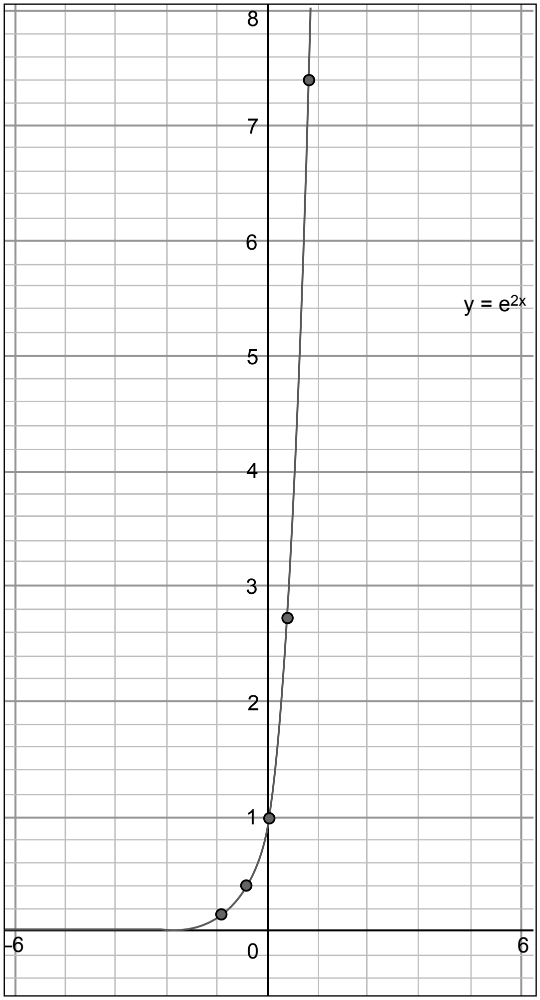

Fig. 1.12: Graph of $y = e^{2x}$

APPLICATIONS (REAL LIFE / INDUSTRIAL)

Flight Path of an Aircraft

An application of Trigonometry to the motion of an aircraft.

Assume that an airplane climbs at a constant angle of 20(degree) from its departure point situated at sea level. It continues to climb at this angle until it reaches its cruise altitude. Suppose that its cruise altitude is 29,580 ft above sea level.

Question 1: What is the distance traveled by the plane from its departure point to its cruise altitude?

Question 2: What is the ground distance traveled by the airplane as it moves from its departure point to its cruise altitude?

Inclined Plane

Work is an important concept in virtually every field of science and engineering. It takes work to move an object; it takes work to move an electron through an electric field; it takes work to overcome the force of gravity; etc.

Let's consider the case where we use an inclined plane to assist in the raising of a 300-pound weight. The inclined plane situated such that one end rests on the ground and the other end rests upon a surface 4 feet above the ground.

Question 3: Suppose that the length of the inclined plane is 12 feet. What is the angle that the plane makes with the ground?

Surveying

Another field of civil engineering is surveying. In particular, one may be interested to investigate how trigonometry can be used to help forest rangers combat fires. Let us suppose that a fire guard observes a fire due south of his Hilltop Lookout location. A second fire guard is on duty at a Watch Tower that is located 11 miles due east of the Hilltop Lookout location. This second guard spots the same fire and measures the bearing (angle) at 2150 from North.

Question: How far away is the fire from the Hilltop Lookout location?

UNIT SUMMARY

In this unit the first topic is devoted to introduce the Concept of angles, measurement of angles in various metrics and their conversions. Second and Third topic deal with T-ratios of Allied angles, Sum, difference formulae and their applications, Product formulae with Transformation of product to sum, difference and vice versa, Ratios of multiple angles and sub-multiple angle. Finally, the fourth topic which gives an insight about Graphs of sin x, cos x, tan x and e^x. Each topic is followed by worked examples along with increasing level of difficulties as per revised Bloom's Taxonomy, similar pattern is adopted while providing the exercise for practice. As an interesting fact some open questions are also advised these Verbal questions will help for assessing conceptual understanding of key terms and concepts. On the other hand Algebraic problems will help students to apply algebraic manipulations, Graphical problems assess students' ability to interpret or produce a graph. Numeric problems require the student perform calculations or computations. Real-World Applications present realistic problem scenarios.

EXERCISES

Multiple Choice Questions

1. If for real values of $x, \cos\theta = x + \frac{1}{x}$, then

(a) θ is an acute angle

(b) θ is a right angle

(c) θ is an obtuse angle

(d) No value of θ is possible

2. The incorrect statement is

(a) $\sin \quad = - -$ (b) $\cos\theta = 1$

(c) $\sec\theta = \dfrac{1}{2}$ (d) $\tan\theta = 20$

3. The equation $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ is only possible when

(a) $x = y$ (b) $x < y$

(c) $x > y$ (d) None of these

4. If $\sin\theta + \operatorname{cosec}\theta = 2$, the value of $\sin^{10}\theta + \operatorname{cosec}^{10}\theta$ is

(a) 10 (b) 2^{10}

(c) 2^{9} (d) 2

5. If $\sin\theta = \dfrac{24}{25}$ and θ lies in the second quadrant, then $\sec\theta + \tan\theta =$

(a) – 3 (b) – 5

(c) – 7 (d) – 9

6. If θ lies in the second quadrant, then the value of $\sqrt{\left(\dfrac{1-\sin\theta}{1+\sin\theta}\right)} + \sqrt{\left(\dfrac{1+\sin\theta}{1-\sin\theta}\right)}$

(a) $2\sec\theta$ (b) $-2\sec\theta$

(c) $2\operatorname{cosec}\theta$ (d) None of these

7. If $\tan\theta + \sec\theta = e^x$, then $\cos\theta$ equals

(a) $\dfrac{\left(e^x + e^{-x}\right)}{2}$ (b) $\dfrac{2}{\left(e^x + e^{-x}\right)}$

(c) $\dfrac{\left(e^x - e^{-x}\right)}{2}$ (d) $\dfrac{\left(e^x - e^{-x}\right)}{\left(e^x + e^{-x}\right)}$

8. $\cos A + \sin(270° + A) - \sin(270° - A) + \cos(180° + A) =$

(a) – 1 (b) 0

(c) 1 (d) None of these

9. If $x = y\cos\dfrac{2\pi}{3} = z\cos\dfrac{4\pi}{3}$, then $xy + yz + zx =$

(a) – 1 (b) 0

(c) 1 (d) 2

10. $\tan 20^\circ + \tan 40^\circ + \sqrt{3} \tan 20^\circ \tan 40^\circ =$

(a) $\dfrac{1}{\sqrt{3}}$ (b) $\sqrt{3}$

(c) $-\dfrac{1}{\sqrt{3}}$ (d) $-\sqrt{3}$

11. $\tan 75^\circ - \cot 75^\circ =$

(a) $2\sqrt{3}$ (b) $2+\sqrt{3}$

(c) $2-\sqrt{3}$ (d) None of these

12. $\dfrac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B} =$

(a) $\tan(A-B)$ (b) $\tan(A \quad B)$

(c) $\cot(A-B)$ (d) $\cot(A+B)$

13. The value of $\cos 12^\circ + \cos 84^\circ + \cos 156^\circ + \cos 132^\circ$ is

(a) $\dfrac{1}{2}$ (b) 1

(c) $-\dfrac{1}{2}$ (d) $\dfrac{1}{8}$

14. The value of $\cos 52^\circ + \cos 68^\circ + \cos 172^\circ$ is

(a) 0 (b) 1

(c) 2 (d) $\dfrac{3}{2}$

15. $\dfrac{\cos 17^\circ + \sin 17^\circ}{\cos 17^\circ - \sin 17^\circ} =$

(a) $\tan 62^\circ$ (b) $\tan 56^\circ$

(c) $\tan 54^\circ$ (d) $\tan 73^\circ$

16. $\cos\dfrac{2\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{8\pi}{15}\cos\dfrac{16\pi}{15} =$

(a) 1/2 (b) 1/4

(c) 1/8 (d) 1/16

17. The value of $\cos^2\dfrac{\pi}{12} + \cos^2\dfrac{\pi}{4} + \cos^2\dfrac{5\pi}{12}$ is

(a) $\dfrac{3}{2}$ (b) $\dfrac{2}{3}$

(c) $\dfrac{3+\sqrt{3}}{2}$ (d) $\dfrac{2}{3+\sqrt{3}}$

18. The value of $\sin\dfrac{\pi}{16}\sin\dfrac{3\pi}{16}\sin\dfrac{5\pi}{16}\sin\dfrac{7\pi}{16}$ is

(a) $\dfrac{1}{16}$ (b) $\dfrac{\sqrt{2}}{16}$

(c) $\dfrac{1}{8}$ (d) $\dfrac{\sqrt{2}}{8}$

19. $\cos^2 76^\circ + \cos^2 16^\circ - \cos 76^\circ \cos 16^\circ =$

(a) – 1/4 (b) 1/2

(c) 0 (d) 3/4

20. $\cos\dfrac{\pi}{7}\cos\dfrac{2\pi}{7}\cos\dfrac{4\pi}{7} =$

(a) 0 (b) $\dfrac{1}{2}$

(c) — (d) $-\dfrac{1}{8}$

21. The value of $\dfrac{\tan 70^\circ - \tan 20^\circ}{\tan 50^\circ} =$

(a) 1 (b) 2

(c) 3 (d) 0

22. $\cos^2\alpha + \cos^2(\alpha+120^\circ) + \cos^2(\alpha-120^\circ)$ is equal to

(a) 3/2 (b) 1

(c) 1/2 (d) 0

23. The value of $\tan 20^\circ + 2\tan 50^\circ - \tan 70^\circ$ is equal to

(a) 1 (b) 0

(c) tan 50° (d) None of these

24. If $\cos(A-B) = \dfrac{3}{5}$ and $\tan A \tan B = 2$, then

(a) $\cos A\cos B = \dfrac{1}{5}$ (b) $\sin A\sin B = -\dfrac{2}{5}$

(c) $\cos A\cos B = -\dfrac{1}{5}$ (d) $\sin A\sin B = -\dfrac{1}{5}$

25. $\sin 12^\circ \sin 48^\circ \sin 54^\circ =$

(a) 1/16 (b) 1/32
(c) 1/8 (d) 1/4

26. $\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\cos\frac{8\pi}{5} =$

(a) 1/16 (b) 0
(c) – 1/8 (d) –1/16

27. $\frac{\cos 12^\circ - \sin 12^\circ}{\cos 12^\circ + \sin 12^\circ} + \frac{\sin 147^\circ}{\cos 147^\circ} =$

(a) 1 (b) – 1
(c) 0 (d) None of these

28. $\tan 20^\circ \tan 40^\circ \tan 60^\circ \tan 80^\circ =$

(a) 1 (b) 2
(c) 3 (d) $\sqrt{3}/2$

29. $\cos 20^\circ \cos 40^\circ \cos 80^\circ =$

(a) 1/2 (b) 1/4
(c) 1/6 (d) 1/8

30. $\sin 36^\circ \sin 72^\circ \sin 108^\circ \sin 144^\circ =$

(a) 1/4 (b) 1/16
(c) 3/4 (d) 5/16

31. If $\sec\theta = 1\frac{1}{4}$, then $\tan\frac{\theta}{2} =$

(a) $\frac{1}{3}$ (b) $\frac{3}{4}$

(c) $\frac{1}{4}$ (d) $\frac{5}{4}$

32. If $\tan\frac{A}{2} = \frac{3}{2}$, then $\frac{1+\cos A}{1-\cos A} =$

(a) -5 (b) 5

(c) $\frac{9}{4}$ (d) $\frac{4}{9}$

33. If $\cos A = \frac{\sqrt{3}}{2}$, then $\tan 3A =$

(a) 0 (b) 1/2
(c) 1 (d) ∞

34. $\sin 4\theta$ can be written as

(a) $4\sin\theta\left(1-2\sin^2\theta\right)\sqrt{1-\sin^2\theta}$ (b) $2\sin\theta\cos\theta\sin^2\theta$

(c) $4\sin\theta-6\sin^3\theta$ (d) None of these

35. $\dfrac{\sin 2A}{1+\cos 2A}\cdot\dfrac{\cos A}{1+\cos A}=$

(a) $\tan\dfrac{A}{2}$ (b) $\cot\dfrac{A}{2}$

(c) $\sec\dfrac{A}{2}$ (d) $\operatorname{cosec}\dfrac{A}{2}$

36. $\dfrac{1}{\tan 3A-\tan A}-\dfrac{1}{\cot 3A-\cot A}=$

(a) $\tan A$ (b) $\tan 2A$

(c) $\cot A$ (d) $\cot 2A$

37. $\operatorname{cosec} A-2\cot 2A\cos A=$

(a) $2\sin A$ (b) $\sec A$

(c) $2\cos A\cot A$ (d) None of these

38. If $\cos 3\theta=\alpha\cos\theta+\beta\cos^3\theta$, then $(\alpha,\beta)=$

(a) $(3,4)$ (b) $(4,3)$

(c) $(-3,4)$ (d) $(3,-4)$

39. $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2=$

(a) $4\cos^2\dfrac{\alpha-\beta}{2}$ (b) $4\sin^2\dfrac{\alpha-\beta}{2}$

(c) $4\cos^2\dfrac{\alpha+\beta}{2}$ (d) $4\sin^2\dfrac{\alpha+\beta}{2}$

40. If $\tan x=\dfrac{b}{a}$, then $\sqrt{\dfrac{a+b}{a-b}}+\sqrt{\dfrac{a-b}{a+b}}=$

(a) $\dfrac{2\sin x}{\sqrt{\sin 2x}}$ (b) $\dfrac{2\cos x}{\sqrt{\cos 2x}}$

(c) $\dfrac{2\cos x}{\sqrt{\sin 2x}}$ (d) $\dfrac{2\sin x}{\sqrt{\cos 2x}}$

41. $1-2\sin^2\left(\dfrac{\pi}{4}+\theta\right)=$

(a) $\cos 2\theta$ (b) $-\cos 2\theta$

(c) $\sin 2\theta$ (d) $-\sin 2\theta$

42. $\dfrac{\sin 3A-\cos\left(\dfrac{\pi}{2}-A\right)}{\cos A+\cos(\pi+3A)}=$

(a) $\tan A$ (b) $\cot A$

(c) $\tan 2A$ (d) $\cot 2A$

43. If $\tan A=\dfrac{1}{2}$, then $\tan 3A=$

(a) $\dfrac{9}{2}$ (b) $\dfrac{11}{2}$

(c) $\dfrac{7}{2}$ (d) $-\dfrac{1}{2}$

44. $\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}=$ (when x lies in IInd quadrant)

(a) $\sin\dfrac{x}{2}$ (b) $\tan\dfrac{x}{2}$

(c) $\sec\dfrac{x}{2}$ (d) $\operatorname{cosec}\dfrac{x}{2}$

45. $(\sec 2A+1)\sec^2 A=$

(a) $\sec A$ (b) $2\sec A$

(c) $\sec 2A$ (d) $2\sec 2A$

Answers of Multiple-Choice Questions

1.	d	**2.**	c	**3.**	a	**4.**	d	**5.**	c
6.	b	**7.**	b	**8.**	b	**9.**	b	**10.**	b
11.	a	**12.**	b	**13.**	c	**14.**	a	**15.**	a
16.	d	**17.**	a	**18.**	b	**19.**	d	**20.**	d
21.	b	**22.**	a	**23.**	b	**24.**	a	**25.**	c
26.	d	**27.**	c	**28.**	c	**29.**	d	**30.**	d
31.	a	**32.**	d	**33.**	d	**34.**	a	**35.**	a
36.	d	**37.**	a	**38.**	c	**39.**	a	**40.**	b
41.	d	**42.**	d	**43.**	b	**44.**	b	**45.**	d

Short and Long Answer Type Questions

1. Convert the following into remaining two units of measurements of an angle.

(i) $60°$ (ii) 3^G

(iii) $\left(\frac{\pi}{6}\right)^R$

2. Convert the following into radians.

(i) $660°$ (ii) $-270°$

(iii) $1440°$ (iv) $0°$

3. Convert the following into degrees.

(i) 17π (ii) $\frac{9\pi}{2}$

(iii) $\frac{82\pi}{6}$ (iv) $\frac{\pi}{3}$

4. Find the value of following T-ratios.

(i) $\sec(-2025°)$ (ii) $\cos(-5\pi+\theta)$

(iii) $\cot\left(\frac{5\pi}{6}\right)$

5. Simplify, $\tan\frac{\pi}{20}\cdot\tan\frac{3\pi}{20}\cdot\tan\frac{5\pi}{20}\cdot\tan\frac{7\pi}{20}\cdot\tan\frac{9\pi}{20}$

6. In a cyclic quadrilateral ABCD, prove that

$\cos(180°-A)-\sin(90°+B)+\cos(180°+C)-\sin(90°+D)=0$

7. Prove that $\dfrac{\tan\left(\frac{\pi}{2}+\theta\right)}{\cot(\pi-\theta)}+\dfrac{\sin(\pi+\theta)}{\sin(2\pi-\theta)}+\dfrac{\cos(2\pi+\theta)}{\sin\left(\frac{\pi}{2}+\theta\right)}=3$

8. Prove that $\tan 660°\cot 1320°+\cot 390°\tan 210°=0$

9. Prove that $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$

10. Evaluate:

(i) $\cot 75^0$ (ii) $\tan\left(\frac{25\pi}{2}\right)$

11. Prove that $\cos\theta=\sqrt{2}\cos\left(\frac{\pi}{4}+\theta\right)+\sin\theta$

12. Prove that $\tan 56° = \dfrac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°}$

13. Prove that $\dfrac{\sin(\alpha+\beta)}{\sin(\alpha+\gamma)} = \cos(\alpha-\gamma) + (\alpha+\gamma)\sin(\alpha-\gamma)$

14. Prove that $\sin[(n+1)x]\cdot\sin[(n+2)x]\cdot\sin[(n+3)x]\cdot\sin[(n+4)x] = \cos x$

15. In a Δ ABC, prove that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

16. Prove that

(i) $\dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \dfrac{1}{\cot A}$ (ii) $\dfrac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \cot A$

17. Prove that $\dfrac{\sin 8\theta \cos\theta - \cos 3\theta \sin 6\theta}{\cos 2\theta \cos\theta - \sin 3\theta \sin 4\theta} = \tan 2\theta$

18. Prove that $\sin 10° \sin 30° \sin 50° \sin 70° = \dfrac{1}{16}$

19. Prove that $\tan A + \cot A = 2\cos ec 2A$ and deduce that $\tan 15° + \cot 15° = 4$

20. Find the value of $\tan\dfrac{\pi}{8}$

21. Show that $\sqrt{2+\sqrt{2+\sqrt{2+2\cos\theta}}} = 2\cos\theta$.

22. Prove that $\sin A \sin(60° + A)\sin(60° - A) = \dfrac{1}{4}\sin 3A$

23. Prove that $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} =$

24. Prove that $\sqrt{\dfrac{1-\cos 2\theta}{2}} = \pm\sin\theta$

25. If $\tan\theta = \dfrac{1}{3}\left(0 < \theta < \dfrac{\pi}{2}\right)$ then prove that $10\sin\theta + 15\cos\theta - 18 = 0$

26. Prove that $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\dfrac{\theta}{2}$

27. Draw the graph of the following.

(i) $y = \sin x\,(0 \le x \le 2\pi)$ (ii) $y = 2\cos x\left(-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}\right)$

28. Draw the graph of $y = 3\sin\frac{x}{2}\left(-2\pi < x < 2\pi\right)$

29. Draw the graph of $y = \tan 2x$

30. Draw the graph of $y = e^{2x}$

Answers of Short and Long Answer Type Questions

1. (i) $66.666...G, \left(\frac{\pi}{3}\right)^R$ (ii) $(2.7)^0, \left(\frac{3\pi}{200}\right)^R$ (iii) $30°, 33.333...G$

2. (i) $\frac{11\pi}{3}$ (ii) $-\frac{3\pi}{2}$ (iii) 8π (iv) 0

3. (i) 3060° (ii) 810° (iii) 2460° (iv) 60°

4. (i) $-\sqrt{2}$ (ii) $-\cos\theta$ (iii) $-\sqrt{3}$

5. 1

10. (i) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (ii) 2 $\sqrt{3}$

20. $\sqrt{2}-1$

KNOW MORE

- Interesting facts about trigonometry.
- Possible reasons to struggle teaching trigonometry to students.
- Why student will like to study trigonometry?
- Need to study Trigonometry.
- Field of education in which trigonometry is used.
- Application of Trigonometry in Calculus.
- Making class more interesting for the students.
- The simplest way to learn trigonometry.
- Why was trigonometry invented?
- Learning of trigonometry intuitively.
- Making of Trigonometry less complicated.

More on Indian Maths

Mini Project

i. Prepare charts showing formulas of multiple and sub multiple trigonometric functions.
ii. Prepare graphical representation for the existence of limits of given functions.

Inquisitiveness and Curiosity Topics

i. A 40 ft ladder leans against the top of a building which is 25 ft tall. Determine the angle the ladder makes with the horizontal. Also determine the distance from the base of the ladder to the building.

ii. A straight trail leads from the Mayur Hotel at elevation 7,000 feet to a scenic overlook at elevation 10,100 feet. The length of the trail is 13,100 feet. What is the inclination angle αin degrees? What is the value of α in radians?

iii. A ray of light moves from a media whose index of refraction is 1.100 to another whose index of refraction is 1.270. The angle of incidence of the ray as it intersects the interface of the two media is 140. Sketch the geometry of the situation and determine the value of the angle of refraction.

iv. One-link planar robots can be used to place pick up and place parts on work table. A one-link planar robot consists of an arm that is attached to a work table at one end. The other end is left free to rotate about the work space. If $l = 4$ cm, sketch the position of the robot and determine the (x, y) coordinates of point p(x,y) for the following values for θ: ($40°$, $2\pi/3$ rad, $-10°$, and $-3\pi/4$ rad).

Apart from the above questions, Trigonometry can be used to roof a house, to make the roof inclined (in the case of single individual bungalows) and the height of the roof in buildings etc. It is being used in naval and aviation industries. It is used in cartography (creation of maps). Also, trigonometry has its applications in satellite systems.

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2 Functions and Limit

UNIT SPECIFICS

This unit elaborately discusses the following topics:

- Definition of function;
- Graph of the function;
- Concept of limits;
- Four standard limits

The applications-based problems are discussed for generating further curiosity and creativity as well as improving problem solving capacity.

Besides giving a large number of multiple-choice questions as well as questions of short and long answer types marked in two categories following lower and higher order of Bloom's taxonomy, assignments through a number of numerical problems, a list of references and suggested readings are given in the unit so that one can go through them for practice.

Based on the content, there is "Know More" section added. This section has been thoughtfully planned so that the supplementary information provided in this part becomes beneficial for the users of the book. This section mainly highlights further teaching and learning related to some interesting facts about Functions and Limits, history of the development of the subject focusing the salient observations, The Historical path of Calculus, role of Limits and Graphs, what are the limits used in Calculus? The simplest way to learn Function and Limit, why was Calculus invented? Learning of Calculus intuitively. Making of Limits less complicated., Teaching Critical Thinking.

On the other hand, suggested Micro projects and brain storming questions create inquisitiveness and curiosity for the topics included in the unit.

RATIONALE

Functions are the major tools for describing the real world in mathematical terms. Slope of a curve at a point, rates of change, area under a curve, accumulations of quantities, derivative is used as a means to demonstrate behaviors of a function. A clear understanding of the concept of a function and familiarity with function notation are important for the study of calculus. To look functions closely graph offers the opportunity to visualize relationships, graphs of functions can be created by plotting points and looking for patterns. A graphing calculator can create graphs rapidly. The limit concept is essential to comprehend the real number system and its distinguishing characteristics. In one sense real numbers can be defined as the numbers that are the limits of convergent sequences of rational numbers. Limits at infinity are useful for describing the end behavior of a function. Limits are useful to calculate derivatives. The derivative is a rate of flow or change, and can be computed based on some limits concepts. Limits are also key to calculating integrals (expressions of areas). The integral calculates the entire area of a region by summing up an infinite number of small pieces of it. Limits are also part of the iterative process. Some successful iterates can get as close as desired to a theoretically exact value.

PRE-REQUISITES

- Knowledge of Exponential and logarithmic functions.
- Familiarity with the use of a calculator
- Familiarity with elementary set theory.
- Familiarity with the algebraic techniques.
- Basic skills for simplifying algebraic expressions and algebraic fractions.
- Expanding brackets.
- Factorising linear and quadratic expressions.
- Solving linear equations and inequalities.
- Solving quadratic equations by factoring, completing the square and applying the quadratic formula.
- Solving simultaneous linear equations.
- Substitution.

UNIT OUTCOMES

List of outcomes of this unit are as follows:

U2-O1: Determine whether or not a correspondence is a function.

U2-O2: Draw the graph of given functions and interpret the Geometrical behaviour.

U 2-O3: Compute limits of functions as the independent variable approaches some finite value or infinity.

U 2-O4: Analyse functions and their graphs as informed by limits.

U 2-O5: Use the concept of limit to calculate the given standard forms.

Unit Outcome	Expected Mapping with Program Outcomes *(1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation)*						
	CO-1	CO-2	CO-3	CO-4	CO-5	CO-6	CO-7
U2-O1	-	-	1	2	2	1	-
U2-O2	-	-	1	2	3	1	-
U2-O3	-	-	2	3	1	1	-
U2-O4	-	-	3	1	2	1	-
U2-O5	-	-	3	2	-	1	-

2.1 FUNCTION

2.1.1 Definition of function

The idea of a function is one of the most important concepts in mathematics. A function is a special kind of correspondence between two sets.

Function can be easily defined with the help of the concept of mapping. Let X and Y be any two non-empty sets. "A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y". Let the correspondence be 'f' then mathematically we write $f : X \to Y$ where $y = f(x), x \in X$ and $y \in Y$. We say that 'y' is the image of 'x' under f (or x is the pre-image of y).

Two things should always be kept in mind:

(i) A mapping $f : X \to Y$ is said to be a function if each element in the set X has its image in set Y. It is also possible that there are few elements in set Y which are not the images of any element in set X.

(ii) Every element in set X should have one and only one image. That means it is impossible to have more than one image for a specific element in set X. *Functions cannot be multi-valued (A mapping that is multi-valued is called a relation from X and Y) e.g.*

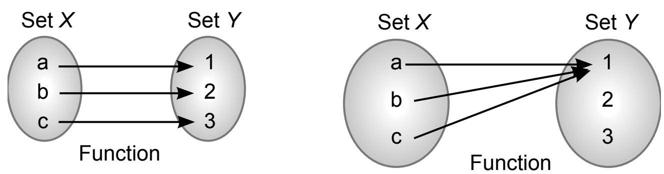


Fig. 2.1: Function

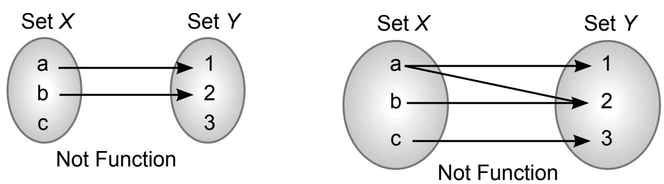


Fig. 2.2: Not a Function

Example 1: Examine each of the following relations given below and state in each case, giving reasons whether it is a function or not?

(i) R = {(2,1), (3,1), (4,2)},

(ii) R = {(2,2), (2,4), (3,3), (4,4)}

(iii) R = {(1,2), (2,3), (3,4), (4,5), (5,6), (6,7)}

Solution:

(i) Since 2, 3, 4 are the elements of R having their unique images, this relation R is a function.

(ii) Since the same first element 2 corresponds to two different images 2 and 4, this relation is not a function.

(iii) Since every element has one and only one image, this relation is a function.

Example 2: Let N be the set of natural numbers. Define a real valued function $f : N \to N$ by $f(x) = 2x + 1$. Using this definition Evaluate $f(1), f(2), f(3)$,

Solution:

$$f(1) = (2 \times 1) + 1 = 3$$

$$f(2) = (2\times 2)+1 = 5$$

$$f(1) = (2\times 3)+1 = 7$$

Example 3: If $f(x) = \begin{cases} x^2; x<0 \\ x; 0\le x<1 \\ \dfrac{1}{x}; x>1 \end{cases}$ Find $f\left(\dfrac{1}{2}\right), f(-2), f(2)$

Solution:

$$f\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$f(-2) = (-2)^2 = 4$$

$$f(2) = \frac{1}{2}$$

2.1.2 Domain, Co-Domain and Range of Function

If a function *f* is defined from a set *A* to set *B* then for $f: A \to B$ set *A* is called the domain of function *f* and set *B* is called the co-domain of function *f*. The set of all *f*-images of the elements of *A* is called the range of function *f*.

In other words, we can say

Domain = All possible values of *x* for which *f*(*x*) exists. Denoted by D_f

Range = For all values of *x*, all possible values of *f*(*x*). Denoted by R_f

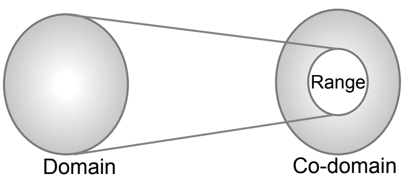


Fig. 2.3: Domain and Codomain

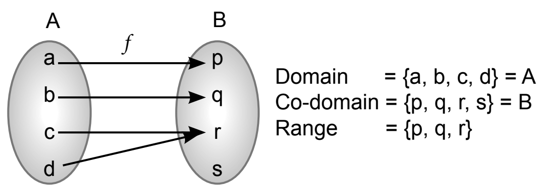


Fig. 2.4: Domain, Codomain and Range

Methods for finding domain and range of function

(i) Domain

(a) Expression under even root (*i.e.*, square root, fourth root etc.) ≥ 0. Denominator ≠ 0.

If domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively then the domain of $f(x) \pm g(x)$ or $f(x).g(x)$ is $D_1 \cap D_2$.

While domain of $\dfrac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{g(x)=0\}$.

Domain of $\left(\sqrt{f(x)}\right) = D_1 \cap \{x : f(x) \geq 0\}$

(ii) Range: Range of $y = f(x)$ is collection of all outputs $f(x)$ corresponding to each real number in the domain.

(a) If domain consists of finite number of points then range is the set of corresponding $f(x)$ values.

(b) If domain is a finite interval, find the least and greatest value for range using monotonicity.

Example 4: Find the domain of the function $f(x) = \dfrac{x^2 + 3x + 5}{x^2 - 5x + 4}$

Solution:

Since $x^2 - 5x + 4 = (x - 4)(x - 1)$, the function f is defined for all real numbers except at $x = 4$ and $x = 1$. Hence the domain of f is R – {1, 4}.

Example 5: Find the domain and the range of the real function f defined by $f(x) = \sqrt{(x-1)}$.

Solution:

Clearly $f(x)$ is defined when $x - 1 \geq 0$ or $x \geq 1$ ie Domain of f is $x \in [1, \infty)$

Clearly Range will be $[0, \infty)$.

More on Function

2.1.3 Some Special Functions their Domain, Range and Graph

- **Constant function:** Let k be a fixed real number. Then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function. The domain of the constant function $f(x) = k$ is the complete set of real numbers and the range of f is the singleton set $\{k\}$. The graph of a constant function is a straight line parallel to x-axis as shown in figure and it is above or below the x-axis according as k is positive or negative. If $k = 0$, then the straight line coincides with x-axis.

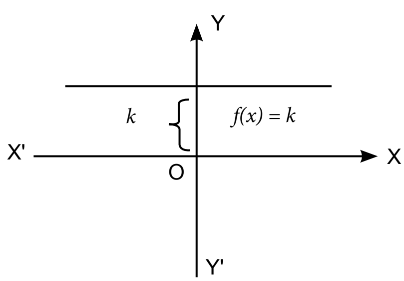


Fig. 2.5: Graph of Constant Function

- **Identity function:** The function defined by $f(x) = x$ for all $x \in R$, is called the identity function on R. Clearly, the domain and range of the identity function is R.

The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with positive direction of x-axis.

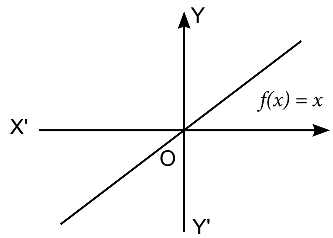


Fig. 2.6: Graph of Identity Function

- **Modulus function:** The function defined by $f(x)=|x|=\begin{cases} x, \text{when} x \geq 0 \\ -x, \text{when} x < 0 \end{cases}$ is called the modulus function. *The domain of the modulus function is the set R of all real numbers and the range is the set of all non-negative real numbers.*

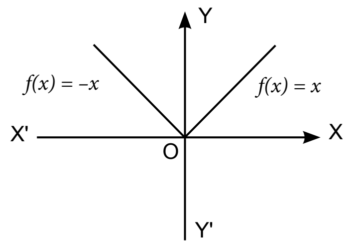


Fig. 2.7: Graph of Modulus Function

- **Greatest integer function:** Let $f(x)=[x]$, where $[x]$ denotes the greatest integer less than or equal to x. ***The domain is R and the range is I***. *e.g.* $[1.1] = 1$, $[2.2] = 2$, $[-0.9] = -1$, $[-2.1] = -3$ etc. The function f defined by $f(x)=[x]$ for all $x \in R$, is called the greatest integer function.

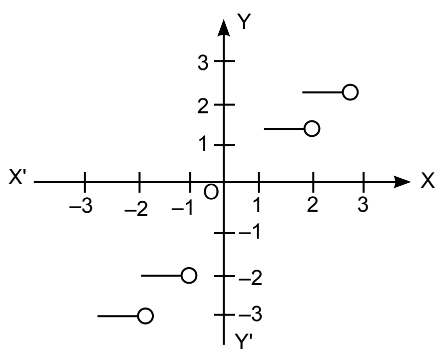


Fig. 2.8: Graph of Greatest Integer Function

- **Signum function:** The function defined by $f\left(x\right)=\begin{cases}\frac{|x|}{x}, x\neq 0\\ 0, x=0\end{cases}$ or $f\left(x\right)=\begin{cases}1, x>0\\ 0, x=0\\ -1, x<0\end{cases}$ is called the signum function. The domain is R and the range is the set {–1, 0, 1}.

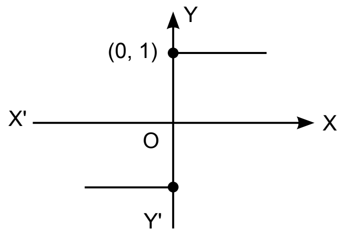


Fig. 2.9: Graph of Signum Function

- **Reciprocal function:** The function that associates each non-zero real number x to be reciprocal $\frac{1}{x}$ is called the reciprocal function. The domain and range of the reciprocal function are both equal to $R-\{0\}$ *i.e.,* the set of all non-zero real numbers. The graph is as shown

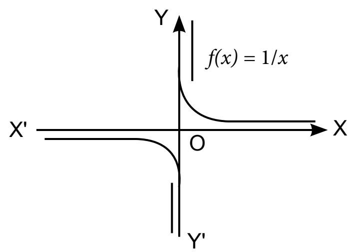


Fig. 2.10: Graph of Reciprocal Function

- **Exponential function:** Let $a\neq 1$ be a positive real number. Then $f:R\to\left(0,\infty\right)$ defined by $f\left(x\right)=a^{x}$ called exponential function. Its domain is R and range is $\left(0,\infty\right)$.
- **Logarithmic function:** The notation ln x is used to denote the natural logarithm of a real number x, the functions ln and Log is $log_{e}x$, the logarithm of x to the base e. When working with functions of a complex variable the notation Log z, with z = $re^{i\theta}$ becomes Log z = ln r + iθ.

  Properties of logarithms: Let m and n be arbitrary positive numbers such that $a>0, a\neq 1$, $b>0,\ b\neq 1$ then

  (1) $\log_{a}a=1, \log_{a}1=0$

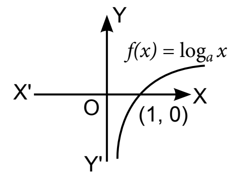
  (2) $\log_{a}b.\log_{b}a=1\Rightarrow\log_{a}b=\frac{1}{\log_{b}a}$

(3) $\log_a(mn) = \log_a m + \log_a n$

(4) $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

(5) $\log_a m^n = n\log_a m$

(6) $a^{\log_a m} = m$



Graph of $f(x) = \log_a x$, when $a > 1$

Fig. 2.11: Graph of Logarithms

- **Even Function:** A function 'f' is said to be an even function if for all $x \in D_f$, $f(-x) = f(x)$.
- **Odd Function:** A function 'f' is said to be an odd function if for all $x \in D_f$, $f(-x) = -f(x)$.

Interval

The set of the numbers between any two real numbers is called **interval**.

(a) Close Interval:

$[a,b] = \{x, a \le x \le b\}$

(b) Open Interval:

(a,b) or $]a,b[= \{x, a < x < b\}$

(c) Semi open or semi close interval:

$[a,b[$ or $[a,b) = \{x; a \le x < b\}$

$]a,b]$ or $(a,b] = \{x; a < x \le b\}$

Graph of a Function

2.2 LIMIT OF A FUNCTION

Sometimes we come across with some functions which do not have definite value corresponding to some particular value of the variable. For example, for the function

$$f(x) = \frac{x^2-4}{x-2},\ f(2) = \frac{4-4}{2-2} = \frac{0}{0}$$

which cannot be determined. Such a form is called an indeterminate form. Some other indeterminate forms are

$$0\times\infty,\ 0^0,\ 1^\infty,\ \infty-\infty,\ \frac{\infty}{\infty},\ \infty^0,\ \frac{0}{0}.$$

Let $y = f(x)$ be a function of x and for some particular value of x say $x = a$, the value of y is indeterminate, then we consider the values of the function at the points which are very near to 'a'. If these values tend to a definite unique number l as x tends to 'a' (either from left or from right) then this unique number l is called the limits of $f(x)$ at $x = a$ and we write it as

$$\lim_{x\to a} f(x) = l$$

Meaning of '$x \to a$' : Let x be a variable and a be a constant. If x assumes values nearer and nearer to 'a' then we can say 'x tends to a' and we write of '$x \to a$'.

By 'x tends to a' we mean that

(i) $x \neq a$

(ii) x assumes values nearer and nearer to 'a' and

(iii) we are not specifying any manner in which x should approach to a. x may approach to a from left or right as shown in figure.

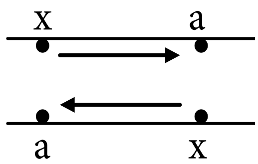


Fig. 2.12: Approach of x to a

2.2.1 Left and Right Limit

If value of a function *f(x)* tend to a definite unique number when x tends to 'a' from left, then this unique number is called left hand limit (LHL) of *f(x)* at $x = a$ and we can write it as

$$f(a-0) \text{ or } \lim_{x\to a^-} f(x) \text{ or } \lim_{x\to a-0} f(x)$$

For evaluation

$$f(a-0) = \lim_{h\to 0} f(a-h)$$

Similarly, we can define right hand limit (RHL) of $f(x)$ at x = a. In this case x tends to 'a' from right. We can write it as

$$f(a+0) \text{ or } \mathit{lim}_{h\to a^+} f(x) \text{ or } \mathit{lim}_{h\to a+0} f(x)$$

For evaluation

$$f(a+0) = \mathit{lim}_{h\to 0} f(a+h)$$

To find Left and Right Limit

(i) For finding right hand limit of the function we write $(x + h)$ in place of x while for left hand limit we write $(x - h)$ in place of x.

(ii) We replace then x by a in the function so obtained.

(iii) Conclusively we find limit $h \to 0$

2.2.2 Existence of Limit

The limit of a function at some point exists only when its left- hand limit and right-hand limit at that point exist and are equal. Thus

$$\mathit{lim}_{x\to a} f(x) \text{ exists, } \lim_{x\to a^-} f(x) = \mathit{lim}_{x\to a^+} f(x) = l$$

where l is called the limit of the function.

Example 6: If $f(x) = \begin{cases} x^2+2, x \ge 1 \\ 2x+1, x<1 \end{cases}$, then $\underset{x\to 1}{Lim} f(x)$ equals –

(a) 1 (b) 2

(c) 3 (d) Does not exist

Solution:

$$\lim_{x\to 1-} f(x) \lim_{h\to 0} = [2(1-h)+1] = 3$$

$$\lim_{x\to 1+} f(0) = \lim_{h\to 0} [(1+h)^2 + 2] = 3$$

Since LHL = RHL, so $\lim_{x\to 1} f(x) = 3$

Example 7: Evaluate $\lim_{x\to -1}\left(\frac{x^2-1}{x^2+3x+2}\right)$

Solution:

$$\text{Limit} = \lim_{x\to -1}\frac{(x-1)(x+1)}{(x+2)(x+1)} = \frac{-1-1}{-1+2} = -2$$

Some Standard Limits

(i) $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$ (ii) $\lim_{x\to 0}\frac{\sin x}{x} = 1$

(iii) $\lim_{x\to 0}\frac{a^x-1}{x} = \log_e a\, a>0$ (iv) $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

Example 8: If $\lim_{x\to 2}\frac{x^n-2^n}{x-2} = 80$, where n is a positive integer, then $n=$

Solution:

$$\lim_{x\to 2}\frac{x^n-2^n}{x-2} = n.2^{n-1} \Rightarrow n.2^{n-1} = 80 \Rightarrow n = 5$$

Example 9: Estimate $\lim_{x\to\sqrt{2}}\frac{x^4-4}{x-\sqrt{2}}$

Solution:

$$\lim_{x\to\sqrt{2}}\frac{x^4-4}{x-\sqrt{2}} = \lim_{x\to\sqrt{2}}\frac{x^4-\left(\sqrt{2}\right)^4}{x-\sqrt{2}}$$

$$= 4\left(\sqrt{2}\right)^{4-1} = 4\left(\sqrt{2}\right)^3 = 4\times\left(2\sqrt{2}\right) = 8\sqrt{2}$$

Example 10: Evaluate $\underset{x\to 16}{lim}\dfrac{x^{\frac{3}{4}}-8}{x-16}$

Solution:

$$\underset{x\to 16}{lim}\frac{x^{\frac{3}{4}}-8}{x-16}=\underset{x\to 16}{lim}\frac{x^{\frac{3}{4}}-(16)^{\frac{3}{4}}}{x-16}$$

$$=\frac{3}{4}(16)^{\frac{3}{4}-1}=\frac{3}{4}(16)^{\frac{-1}{4}}=\frac{3}{4}\left(\frac{1}{16}\right)^{\frac{1}{4}}$$

$$=\frac{3}{4}\times\frac{1}{2}=\frac{3}{8}$$

Example 11: Estimate $\lim\limits_{x\to -1}\dfrac{x^{17}+1}{x^{9}+1}$

Solution:

$$\lim_{x\to -1}\frac{x^{17}+1}{x^{9}+1}=\lim_{x\to -1}\frac{x^{17}-(-1)^{17}}{x^{9}-(-1)^{9}}$$

$$=\frac{\lim\limits_{x\to -1}\dfrac{x^{17}-(-1)^{17}}{x-1}}{\lim\limits_{x\to -1}\dfrac{x^{9}-(-1)^{9}}{x-1}}=\frac{17(-1)^{16}}{9(-1)^{8}}$$

$$=\frac{17}{9}$$

Example 12: Evaluate $\lim\limits_{x\to 0}\dfrac{\sin 3x}{5x}$

Solution:

$$\lim_{x\to 0}\frac{\sin 3x}{5x}=\lim_{x\to 0}\left(\frac{\sin 3x}{5x}\times\frac{3}{3}\right)$$

$$=\frac{3}{5}\lim_{3x\to 0}\frac{\sin 3x}{3x}\qquad [\because 3x\to 0\, as\, x\to 0]$$

$$=\frac{3}{5}\times 1=\frac{3}{5}\qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

Example 13: Evaluate $\lim\limits_{x\to 0}\dfrac{5x\ \cos x-2\sin x}{7x+5sinx}$

Solution:

$$\lim_{x\to 0}\frac{5x\ \cos x-2\sin x}{7x+5sinx}=\lim_{x\to 0}\frac{\dfrac{5x\ \cos x-2\sin x}{x}}{\dfrac{7x+5sinx}{x}}$$

$$=\lim_{x\to 0}\frac{\dfrac{5x\ \cos x}{x}-\dfrac{2sinx}{x}}{\dfrac{7x}{x}+\dfrac{5sinx}{x}}=\lim_{x\to 0}\frac{5cosx-\dfrac{2sinx}{x}}{7+\dfrac{5sinx}{x}}$$

$$=\frac{5\lim\limits_{x\to 0}cosx-\dfrac{2\lim\limits_{x\to 0}sinx}{x}}{\lim\limits_{x\to 0}7+5\lim\limits_{x\to 0}\dfrac{sinx}{x}}$$

$$=\frac{5(1)-2(1)}{7+5(1)}=\frac{3}{15}=\frac{1}{4}$$

Example 14: Estimate $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x}$

Solution:

$$\lim_{x\to 0}\frac{1-\cos 2x}{x}=\lim_{x\to 0}\frac{x.2\sin^2 x}{x^2}=2.\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\lim_{x\to 0}x=0$$

Example 15: Evaluate $\lim\limits_{\theta\to 0}\dfrac{\cos ec\theta-\cot\theta}{\theta}$

Solution:

$$\lim_{\theta\to 0}\frac{\cos ec\theta-\cot\theta}{\theta}=\lim_{\theta\to 0}\frac{\dfrac{1}{\sin\theta}-\dfrac{\cos\theta}{\sin\theta}}{\theta}$$

$$=\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta\sin\theta}=\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta\sin\theta}\times\frac{1+\cos\theta}{1+\cos\theta}$$

$$=\lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta\sin\theta(1+\cos\theta)}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta(1+\cos\theta)}$$

$$=\frac{\lim\limits_{x\to 0}\dfrac{\sin\theta}{\theta}}{\lim\limits_{x\to 0}(1+\cos\theta)}=\frac{1}{1+1}=\frac{1}{2}$$

Example 16: Assess $\lim\limits_{x\to 0}\dfrac{\sec 5x-\sec 3x}{\sec 3x-\sec x}$

Solution:

$$\lim_{x\to 0}\frac{\sec 5x-\sec 3x}{\sec 3x-\sec x}=\lim_{x\to 0}\frac{\cos 3x-\cos 5x}{\cos x-\cos 3x}\cdot\frac{\cos x}{\cos 5x}$$

$$=\lim_{x\to 0}\frac{-2\sin 4x\sin x}{-2\sin 2x\sin x}\cdot\frac{\cos x}{\cos 5x}$$

$$=\lim_{x\to 0}\frac{\sin 4x}{\sin 2x}\cdot\frac{4x}{4x}\cdot\frac{\cos x}{\cos 5x}=\frac{2\lim\limits_{4x\to 0}\dfrac{\sin 4x}{4x}}{\lim\limits_{2x\to 0}\dfrac{\sin 2x}{2x}}\cdot\lim_{x\to 0}\frac{\cos x}{\cos 5x}$$

$$=\frac{2(1)}{1}\cdot\frac{1}{1}=2$$

Example 17: Evaluate $\lim\limits_{x\to 0}\dfrac{2^{x+3}-8}{x}$

Solution:

$$\lim_{x\to 0}\frac{2^{x+3}-8}{x}=\lim_{x\to 0}\frac{2^x\cdot 2^3-2^3}{x}=\lim_{x\to 0}\frac{2^3\left(2^x-1\right)}{x}$$

$$=8\lim_{x\to 0}\frac{\left(2^x-1\right)}{x}=8\times\log_e 2=\log_e 256$$

Example 18: Evaluate $\lim\limits_{x\to e}\dfrac{1-\log_e x}{e-x}$

Solution: let, $\log_e x=y$

$\therefore x=e^y$ (by the definition of logarithm) and $x\to e\Rightarrow y\to 1$

$$\lim_{x\to e}\frac{1-\log_e x}{e-x}=\lim_{y\to 1}\frac{1-y}{e-e^y}=\lim_{y\to 1}\frac{1}{e}\left(\frac{y-1}{e^{y-1}-1}\right)$$

$$=\frac{1}{e}\lim_{y\to 1}\left(\frac{1}{\dfrac{e^{y-1}-1}{y-1}}\right)=\frac{1}{e}\left(\frac{\lim\limits_{y\to 1}1}{\lim\limits_{y-1\to 0}\dfrac{e^{y-1}-1}{y-1}}\right)\left\{\because y\to 1\Rightarrow y-1\to 0\right.$$

$$=\frac{1}{e}\times\frac{1}{1}\left\{\because \lim_{x\to 0}\frac{e^x-1}{x}=\log_e e=1\right.$$

$$=\frac{1}{e}$$

Example 19: Evaluate $\lim\limits_{x \to 0} \dfrac{e^{\sin x} - 1}{x} =$

Solution:

$$\lim_{x \to 0} \frac{e^{\sin x} - 1}{x} = \lim_{x \to 0} \frac{e^{\sin x} - 1}{\sin x} \times \frac{\sin x}{x}$$

$$\underset{\lim \sin x \to 0}{=} \frac{e^{\sin x} - 1}{\sin x} \times \lim_{x \to 0} \frac{\sin x}{x} = 1 \times 1 = 1 \qquad \{\because x \to 0 \text{ then } \sin x \to 0$$

Example 20: Evaluate $\lim\limits_{x \to 0} (1+3x)^{\frac{2}{x}}$

Solution:

$$\lim_{x \to 0} (1+3x)^{\frac{2}{x}} = \lim_{x \to 0} (1+3x)^{\frac{1}{3x} \times 6}$$

$$= \lim_{3x \to 0} \left[(1+3x)^{\frac{1}{3x}} \right]^6 \quad \{ since\, 3x \to 0\, as\, x \to 0$$

$$= \left[\lim_{3x \to 0} (1+3x)^{\frac{1}{3x}} \right]^6$$

$$= e^6$$

Example 21: Evaluate $\lim\limits_{x \to \infty} \left(1 + \dfrac{2}{3x}\right)^{5x}$

Solution:

$$\lim_{x \to \infty} \left(1 + \frac{2}{3x}\right)^{5x} = \lim_{x \to \infty} \left(1 + \frac{2}{3x}\right)^{\frac{3x}{2} \times \frac{10}{3}} = \left[\lim_{\frac{2}{3x} \to 0} \left(1 + \frac{2}{3x}\right)^{\frac{3x}{2}} \right]^{\frac{10}{3}} \left\{ x \to \infty then \frac{1}{x} \to 0 \right. \quad = e^{\frac{10}{3}}$$

Example 22: Calculate $\lim\limits_{x \to 0} \dfrac{x \log(x+1)}{1 - \cos x}$

Solution:

$$\lim_{x \to 0} \frac{x\log(x+1)}{1 - \cos x} = \lim_{x \to 0} \frac{\frac{x\log(x+1)}{x^2}}{\frac{1 - \cos x}{x^2}}$$

$$= \lim_{x \to 0} \frac{\frac{1}{x} \log(x+1)}{\frac{2\sin^2 \frac{x}{2}}{x^2}} = \frac{\lim\limits_{x \to 0} \frac{1}{x} \log(x+1)}{\lim\limits_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{x^2}}$$

$$= \frac{\lim_{x\to 0} \log(x+1)^{\frac{1}{x}}}{\lim_{x\to 0} \frac{2\left(\sin\frac{x}{2}\right)^2}{x^2 \times \frac{4}{4}}} = \frac{\log \lim_{x\to 0}(x+1)^{\frac{1}{x}}}{\frac{1}{2}\lim_{x\to 0} \frac{\left(\sin\frac{x}{2}\right)^2}{\left(\frac{x}{2}\right)^2}}$$

$$= \frac{\log \lim_{x\to 0}(x+1)^{\frac{1}{x}}}{\frac{1}{2}\left[\lim_{x\to 0} \frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}\right]^2} = \frac{\log e}{\frac{1}{2}(1)} = 2$$

APPLICATIONS (REAL LIFE / INDUSTRIAL)

Compound interest:

Example 1: An investor deposits Rs 20,000 in Mutual Funds, at 3%. How much is the investment worth at the end of 1 yr., if interest is compounded:

a) annually? b) semiannually) quarterly?

Scaling stress factors:

Example 2: In psychology a process called scaling is used to attach numerical ratings to a group of life experiences. In the table below, various events have been rated on a scale from 1 to 100 according to the stress levels.

Table 2.1: Events and their Scale of Impact

Event	Scale of Impact
Death of spouse	90
Divorce	63
Convicted	53
Marriage	30
Unemployed	57
Pregnancy	40
Loan over Rs 10,000	32
Change in schools	30
Loan less than Rs 10,000	20
Festival Celebration	12

(a) Does the above table represent a function? Why or why not?

(b) What are the inputs? What are the outputs?

Nerve impulse speed:

Example 3: Impulses in nerve fibers travel at a speed of 283 ft/sec. The distance D, in feet, traveled in t sec is given by D=283t. How long would it take an impulse to travel from the brain to the toes of a person who is 5.5 ft tall?

Soda, Snack, or Stamp Machine:

Example 4: The user puts in money, punches a specific button, and a specific item drops into the output slot. The function rule is the product price. The input is the money combined with the selected button. The output is the product, sometimes delivered along with coins in change, if the user has entered more money than required item.

Rate of Change:

Example 5: An athlete begins he normal practice for the next marathon during the evening. At 6:15 pm he starts to run and leaves his home. At 7:45 pm, the athlete finishes the run at home and has run a total of 6.5 miles. How fast was his average speed over the course of the run?

Real life Model with Multiple Equations:

Example 6: Initially, trains A and B are 305miles away from each other. Train A is traveling towards B at 40 miles per hour and train B is traveling towards A at 70miles per hour. At what time will the two trains meet? At this time how far did the trains travel?

Examples of limits:

Example 7: Measuring the temperature of a hot rod sunk in a chilled glass of water is a limit. Other examples, like measuring the strength of an electric, magnetic or gravitational field. In the case of limits, when we relate it to infinity it means how the numbers behave as they are getting larger or a series, where new numbers are continuously added.

Chemical Reaction:

Example 8: Chemical reaction started in a beaker in which two different compounds react to form a new compound. Now as time approaches infinity, the quantity of the new compound formed is a limit.

Example 9: A jet airplane averages 830 km/hr. to fly the 3000km between the city A & B. how many hours the flight takes?

UNIT SUMMARY

In this unit the first topic is devoted to present the Concept of Functions, types of the functions, Graphs of the functions and its related properties. In the second topic concept of the Limit along with various types are presented. Each topic is followed by worked examples along with increasing level of difficulties as per revised Bloom's Taxonomy, similar pattern is adopted while providing the exercise for practice. New examples designed to reinforce the main concepts and applications of limits and derivatives, so that students have some familiarity with the derivative as an analytical tool, as opposed to a formula to be memorized. Mathematical equations can serve as models of many kinds of applications. As an interesting fact some open questions are also advised these Verbal questions will help for assessing conceptual understanding of key terms and concepts. On the other hand. Algebraic problems will help students to apply algebraic manipulations, Graphical problems assess students' ability to interpret or produce a graph. Numeric problems require the student perform calculations or computations. Real-World Applications present realistic problem scenarios.

EXERCISES

Multiple Choice Questions

1. If $f(x)=\dfrac{1}{\sqrt{x+2\sqrt{2x-4}}}+\dfrac{1}{\sqrt{x-2\sqrt{2x-4}}}$ for $x>2$, then $f(11)=$ ....

(a) 7/6 (b) 5/6
(c) 6/7 (d) 5/7

2. Domain of the function $f(x)=\dfrac{x^2-3x+2}{x^2+x-6}=$ ....

(a) $\{x: x\in R,\ x\neq 3\}$
(b) $\{x: x\in R,\ x\neq 2\}$
(c) $\{x: x\in R\}$
(d) $\{x: x\in R,\ x\neq 2, x\neq -3\}$

3. The domain of $f(x)=\dfrac{1}{x^3-x}=$ ....

(a) R – {–1,0,1} (b) R
(c) R – {0,1} (d) None of these

4. The range of function $f(x)=\dfrac{x^2}{1+x^2}=$ ....

(a) R – {1} (b) $R^+ \cup \{0\}$
(c) [0, 1] (d) None of these

5. If the domain of function $f(x)=x^2-6x+7$ is $(-\infty,\infty)$, then the range of function is

(a) $(-\infty,\infty)$ (b) $[-2,\infty)$
(c) $(-2,3)$ (d) $(-\infty,-2)$

6. Domain of the function $f(x)=\dfrac{1}{\sqrt{x+2}}=$ ....

(a) R (b) $(-2, \infty)$
(c) $[2, \infty]$ (d) $[0, \infty]$

7. Domain of the function $\sqrt{2-x}-\dfrac{1}{\sqrt{9-x^2}}=$ ....

(a) $(-3, 1)$ (b) $[-3, 1]$
(c) $(-3, 2]$ (d) $[-3, 1)$

8. Domain of the function $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}=$ ....

(a) $(-1, 1)$ (b) $(-1, 1)-\{0\}$
(c) $[-1, 1]$ (d) $[-1, 1]-\{0\}$

9. The domain of the function $f(x)=\sqrt{x-x^2}+\sqrt{4+x}+\sqrt{4-x}$ i= ....

(a) $[-4, \infty)$ (b) $[-4, 4]$
(c) $[0, 4]$ (d) $[0, 1]$

Problems based on limit of function

10. $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1+x}-\sqrt{1-x}}=$ ....

(a) 1/2 (b) 2
(c) 1 (d) 0

11. $\lim\limits_{x\to 0}\dfrac{\tan 2x-x}{3x-\sin x}=$ ....

(a) 2/3 (b) 1/3
(c) 1/2 (d) 0

12. $\lim\limits_{x\to\infty}\left(\dfrac{x+2}{x+1}\right)^{x+3}=$ ....

(a) 1 (b) e
(c) e^2 (d) e^3

13. $\lim\limits_{x\to 0}\dfrac{\sin x^{o}}{x}$ is = ...

(a) 1 (b) π
(c) x (d) $\pi/180$

14. If $f(x)=\begin{cases} x-1, x<0 \\ 1/4, x=0 \\ x^2, x>0 \end{cases}$ then $\lim\limits_{x\to 0} f(x)=$ ....

(a) 0 (b) 1
(c) -1 (d) Does not exist

15. If $f(x) = \begin{cases} 4x, & x < 0 \\ 1, & x = 0 \\ 3x^2, & x > 0 \end{cases}$, then $\lim\limits_{x \to 0} f(x) =$ ....

(a) 0 (b) 1

(c) 3 (d) Does not exist

16. $\lim\limits_{x \to \infty} \sin x =$ ....

(a) 1 (b) 0

(c) ∞ (d) Does not exist

17. $\lim\limits_{x \to 0} \sin \dfrac{1}{x} =$ ....

(a) 0 (b) 1

(c) ∞ (d) Does not exist

18. $\lim\limits_{x \to 0} x \sin \dfrac{1}{x} =$ ....

(a) 1 (B) 0

(c) ∞ (d) None of these

19. $\lim\limits_{x \to a} \dfrac{\sqrt{x} + \sqrt{a}}{x + a} =$ ....

(a) $\sqrt{x}$ (b) $\sqrt{a}$

(c) $\dfrac{1}{\sqrt{a}}$ (d) $\dfrac{1}{\sqrt{x}}$

20. $\lim\limits_{x \to \infty} \left(1 + \dfrac{3}{2x}\right)^{\frac{2x}{3}} =$ ....

(a) $\dfrac{1}{e}$ (B) e

(c) $e^{\frac{2}{3}}$ (d) $e^{\frac{3}{2}}$

Answers of Multiple-Choice Questions									
1.	c	**2.**	d	**3.**	a	**4.**	c	**5.**	b
6.	b	**7.**	c	**8.**	d	**9.**	d	**10.**	c
11.	c	**12.**	b	**13.**	d	**14.**	d	**15.**	a
16.	d	**17.**	d	**18.**	b	**19.**	c	**20.**	b

Short and Long Answer Type Questions

1. If $f(x) = \log x$ then prove that

$$f(x \cdot y) = f(x) + f(y) \text{ and } f\left(\frac{x}{y}\right) = f(x) - f(y)$$

2. If $f(x) = \tan x$ then prove that

$$f(2x) = \frac{2f(x)}{1 - [f(x)]^2}$$

3. Evaluate.

(i) $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x^2 - 4}$

(ii) $\lim\limits_{x \to 2} \dfrac{x^3 - x^2 - 5x + 6}{x^2 - 6x + 8}$

(iii) $\lim\limits_{x \to a} \dfrac{\sqrt{2a - x} - \sqrt{x}}{a - x}$

4. Evaluate.

(i) $\lim\limits_{x \to 1^+} \dfrac{\sqrt{x^2 - 1} + \sqrt{x - 1}}{\sqrt{x^2 - 1}}$

(ii) $\lim\limits_{x \to 7} \dfrac{\sqrt{9 + x} - 4}{\sqrt{8 - x} - 1}$

(iii) $\lim\limits_{x \to \infty} \dfrac{2x^2 - 3x + 5}{4x^2 - 5x - 9}$

5. Evaluate.

(i) $\lim\limits_{x \to -3} \dfrac{x^3 + 27}{x + 3}$

(ii) $\lim\limits_{x \to a} \dfrac{\sqrt[3]{x} - \sqrt[3]{a}}{\sqrt[7]{x} - \sqrt[7]{a}}$

(iii) $\lim\limits_{x \to 2} \dfrac{x^7 - 128}{x^{\frac{1}{2}} - 2^{\frac{1}{2}}}$

6. If $\lim\limits_{x \to 3} \dfrac{x^n - 3^n}{x - 3} = 27$ then find n.

7. Calculate.

(i) $\lim\limits_{\theta \to 0} \dfrac{\sin(a\theta)}{\sin(b\theta)}$

(ii) $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{2 - \sec^2 x}{1 - \tan x}$

(iii) $\lim\limits_{\theta \to 0} \dfrac{\operatorname{cosec}\theta - \cot\theta}{\theta}$

8. Evaluate.

(i) $\lim\limits_{\to} \dfrac{5^{3x} \quad 3^{2x}}{}$ (ii) $\lim\limits_{x\to 0} \dfrac{12^x - 4^x - 3^x + 1}{x^2}$

(iii) $\lim\limits_{x\to 0} \dfrac{e^x - \sin x - 1}{x}$

9. Estimate.

(i) $\lim\limits_{x\to 0} \left(1+\dfrac{3x}{4}\right)^{\frac{6}{x}}$ (ii) $\lim\limits_{x\to 0} \left(1+\dfrac{4x}{7}\right)^{\frac{3}{2x}}$

(iii) $\lim\limits_{x\to 0} \left(\dfrac{x+2}{x+1}\right)^{x}$

10. Evaluate.

(i) $\lim\limits_{x\to 0} \dfrac{\cos 3x - \cos 2x}{x^2}$ (ii) $\lim\limits_{x\to 0} \dfrac{\sqrt{2} - \sqrt{1+\cos x}}{\sin^2 x}$

(iii) $\lim\limits_{x\to 0} \dfrac{6^x - 3^x - 2^x - 1}{x^2}$

Answers of Short and Long Answer Type Questions

3. (i) 3 (ii) $-\dfrac{3}{2}$ (iii) $\dfrac{1}{\sqrt{a}}$

4. (i) $\dfrac{\sqrt{2}+1}{\sqrt{2}}$ (ii) $-\dfrac{1}{4}$ (iii) $\dfrac{1}{2}$

5. (i) 27 (ii) $\dfrac{7}{3}(a)^{\frac{4}{21}}$ (iii) $896\sqrt{2}$

6. 3

7. (i) $\dfrac{a}{b}$ (ii) 2 (iii) $\dfrac{1}{2}$

8. (i) $\log\left(\dfrac{125}{9}\right)$ (ii) $\log 3.\log 4$ (iii) 0

9. (i) $e^{\frac{9}{2}}$ (ii) $e^{\frac{6}{7}}$ (iii) e

10. (i) $-\dfrac{5}{2}$ (ii) $\dfrac{1}{4\sqrt{2}}$ (iii) $\log 3 \cdot \log 2$

KNOW MORE

- The Historical path of Calculus.
- Limits and Graphs.
- Limits and Factoring.
- What are the limits used in Calculus?
- Teaching and Learning Functions.
- Shift to Online Teaching.
- The simplest way to learn Function and Limit.
- Why was Calculus invented?
- Learning of Calculus intuitively.
- Making of Limits less complicated.
- Online Education Tools for Teachers.
- Teaching Critical Thinking
- STEM Education.

Mini Project

i. Use Graphing calculator to plot the graph of functions showing Engineering applications.
ii. Collect set of problems based on functions and limit with real world applications.

Inquisitiveness and Curiosity Topics

i. How function values(outputs) change as input values change. Basic to this study is the notion of a limit.
ii. What is the role of infinity with respect to some functions, one may answer this question by the concept of Limits.
iii. When an engineer designs a new car, model the gasoline through the car's engine with small intervals called a mesh, this problem reduces to simply functions known as polynomials. These approximations always use limits.
iv. How do we determine a linear function from a table and graph?
v. How do we determine whether the given equation is a linear equation or a nonlinear equation?
vi. How are limits used or applied to daily life or to the real-world problems?
vii. What happens when the independent variable becomes very large? One may find the answer, as 't' becomes very large, f(t) becomes very small. In this case f(t) approaches zero as 't' goes to infinity for example damping of simple harmonic motion.

Apart from the above questions, other applications of function and limit can be corelated with problems based on Speed Limit, Vehicle capacity, Limit of food we intake, Limit of using the internet, Limit of medicine amount we intake.

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3 Differential Calculus

UNIT SPECIFICS

This unit elaborately discusses the following topics:

- Differentiation by definition of algebraic, trigonometric, exponential and logarithmic functions;
- Differentiation of sum;
- Product and quotient of functions;
- Differentiation of function of a function;
- Differentiation of trigonometric and inverse trigonometric functions;
- Logarithmic differentiation;
- Exponential functions.

The applications-based problems are discussed for generating further curiosity and creativity as well as improving problem solving capacity.

Besides giving a large number of multiple-choice questions as well as questions of short and long answer types marked in two categories following lower and higher order of Bloom's taxonomy, assignments through a number of numerical problems, a list of references and suggested readings are given in the unit so that one can go through them for practice.

Based on the content, there is "Know More" section added. This section has been thoughtfully planned so that the supplementary information provided in this part becomes beneficial for the users of the book. This section mainly highlights further teaching and learning related to some interesting facts about development of Differential Calculus, why study Differentiation? how calculus is used in our daily lives, Limits and Differentiation, why mathematical thinking is valuable in daily life, the simplest way to learn Differentiation, why was Differential Calculus invented? Learning of Differential Calculus intuitively, making of Differentiation less complicated., STEM Education.

On the other hand, suggested Micro projects and brain storming questions create inquisitiveness and curiosity for the topics included in the unit.

RATIONALE

Calculus is the language of engineers, scientists, and economists. Calculus is a branch of Mathematics that calculates how matter, particles and heavenly bodies actually move. With calculus, we can find how the changing conditions of a system affect us, we can control a system. We always come across quantities which change with respect to each other and these rates of change can often be expressed as derivatives. To determine the maximum and minimum values of particular functions for example cost, strength, amount of material used in a building etc. we use the concept of derivative which further help us to modelling the behavior of moving objects. Differential Calculus makes it possible to solve problems as diverse as tracking the position of a space shuttle and other similar problems. Computers have now become a valuable tool for cracking Differential calculus problems that were once hard nut to crack. Calculus has an elegant beauty that leads mathematicians to view it as a state of art.

PRE-REQUISITES

- Functions and their graphs
- Transforming a function
- Trigonometric functions.
- Trigonometric identities
- Coordinate geometry
- The binomial theorem.
- Limits and continuity.
- Familiarity with the algebraic techniques.
- Substitution.

UNIT OUTCOMES

List of outcomes of this unit are as follows:

U3-O1: Compute derivative of functions of one variable.

U3-O2: Compute derivatives of given algebraic, trigonometric, inverse trigonometric, exponential, and logarithmic functions.

U 3-O3: Use the derivative of a function to determine the properties of the graph of the function.

U 3-O4: Demonstrate the ability to graphically analyse and interpret geometrically the derivative of given functions.

U 3-O5: Use the concept of derivative in real world situations?

Unit Outcome	Expected Mapping with Program Outcomes *(1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation)*						
	CO-1	CO-2	CO-3	CO-4	CO-5	CO-6	CO-7
U3-O1	-	-	1	1	1	1	-
U3-O2	-	1	2	2	2	2	-
U3-O3	-	1	3	3	2	1	-
U3-O4	-	1	3	3	2	1	-
U3-O5	-	1	3	2	3	2	-

3.1 DERIVATIVE OF FUNCTION AT A POINT

Suppose f is a real valued function and a is a point in its domain of definition. The derivative of f at a is defined by

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Provided this limit exists. Derivative of f (x) at a is denoted by $f'(a)$ Observe that $f'(a)$ quantifies the change in $f(x)$ at a with respect to x.

Example 1: Find the derivative at $x = 2$ of the function $f(x) = 3x$

Solution: $f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{3(2+h) - 3(2)}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3$

3.1.1 Derivative of Function

For a given function f we can find the derivative at every point. If the derivative exists at every point, it defines a new function called the derivative of f. Formally, we define derivative of a function as follows.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

wherever the limit exists is defined to be the derivative of f at x and is denoted by $f'(x)$. This definition of derivative is also called the first principle of derivative. Thus

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

3.1.2 Differentiation of Some Standard Function by Definition

Example 2: $y = f(x) = x^n$ find the derivative by definition

Solution: We know

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - (x)^n}{h} = \lim_{x+h \to x} \frac{(x+h)^n - (x)^n}{(x+h) - x}$$

$$= nx^{n-1} \quad ; \text{Apply } \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Example 3: $y = f(x) = sinx$ find the derivative by definition

Solution:

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{x+h+x}{2}\right) \cdot \sin\left(\frac{x+h-x}{2}\right)}{h} = \lim_{h \to 0} \frac{\cos\left(x + \frac{h}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$$= \lim_{h \to 0} \cos\left(x + \frac{h}{2}\right) \cdot \lim_{\frac{h}{2} \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} = \cos x \cdot 1 = \cos x \cdot$$

Apply the limit $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Example 4: $y = f(x) = cosx$ find the derivative by definition

Solution:

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

$$= \lim_{h\to 0}\frac{-2\sin\left(\frac{x+h+x}{2}\right)\cdot\sin\left(\frac{x+h-x}{2}\right)}{h} = \lim_{h\to 0}\frac{-\sin\left(x+\frac{h}{2}\right)\cdot\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}$$

$$= -\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\cdot\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} = -\sin x\cdot 1 = -\sin x$$

Apply the limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Example 5: $y = f(x) = tanx$ find the derivative by definition

Solution:

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}}{h} = \lim_{h\to 0}\frac{\sin(x+h)\cdot\cos x-\sin x\cdot\cos(x+h)}{h\cdot\cos x\cdot\cos(x+h)}$$

$$= \lim_{h\to 0}\frac{\sin(x+h-x)}{h\cdot\cos x\cdot\cos(x+h)} = \lim_{h\to 0}\frac{\sinh}{h}\cdot\lim_{h\to 0}\frac{1}{\cos x\cdot\cos(x+h)}$$

$$= 1\cdot\frac{1}{\cos x\cdot\cos x} = \frac{1}{\cos^2 x} = \sec^2 x$$ Apply the limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Example 6: $y = f(x) = e^x$ find the derivative by definition

Solution:

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x\left(e^h-1\right)}{h} = e^x\lim_{h\to 0}\frac{\left(e^h-1\right)}{h} = e^x\cdot\log_e e = e^x\cdot 1 = e^x$$

Apply the limit $\lim_{h\to 0}\frac{e^h-1}{h} = 1$

Example 7: $y = f\left(x\right) = log_e x$ find the derivative by definition

Solution:

$$\frac{dy}{dx} = f'\left(x\right) = \lim_{h\to 0}\frac{f\left(x+h\right)-f\left(x\right)}{h} = \lim_{h\to 0}\frac{\log(x+h)-\log x}{h}$$

$$= \lim_{h\to 0}\frac{\log\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0}\frac{1}{h}\cdot\log\left(\frac{x+h}{x}\right) = \lim_{h\to 0}\log\left(\frac{x+h}{x}\right)^{\frac{1}{h}}$$

$$= \log\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h}}\right] = \log\left[\left\{\lim_{\frac{h}{x}\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right\}^{\frac{1}{x}}\right] = \log_e e^{\frac{1}{x}} = \frac{1}{x}\log_e e = \frac{1}{x}$$

(1) Differentiation of algebraic functions:

(i) $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ (ii) $\frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{n}{x^{n+1}}$

(iii) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$ (iv) $\frac{d}{dx}\left(k\right) = 0$ where k is constant

(2) Differentiation of trigonometric functions:

(i) $\frac{d}{dx}\sin x = \cos x$ (ii) $\frac{d}{dx}\cos x = -\sin x$

(iii) $\frac{d}{dx}\tan x = \sec^2 x$ (iv) $\frac{d}{dx}\sec x = \sec x\tan x$

(v) $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x\cot x$ (vi) $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$

(3) Differentiation of logarithmic and exponential functions:

(i) $\frac{d}{dx}\log x = \frac{1}{x}$, for $x> 0$ (ii) $\frac{d}{dx}e^x = e^x$

(iii) $\frac{d}{dx}a^x = a^x\log a$, for $a> 0$

(iv) $\frac{d}{dx}\log_a x = \frac{1}{x\log a}$, for $x> 0$, $a> 0$, $a\neq 1$

(4) Differentiation of inverse trigonometric functions:

(i) $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$

(ii) $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$, for $-1 < x < 1$

(iii) $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$

(iv) $\frac{d}{dx}\text{cosec}^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$

(v) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$, for $x \in R$

(vi) $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$, for $\mathrm{x} \in \mathrm{R}$

3.2 ALGEBRA OF DERIVATIVE OF FUNCTIONS

Since the definition of derivatives involve limits in a rather direct fashion, we assume the rules for derivatives to follow carefully that of limits. These are collected in the following results.

Let f and g be two functions such that their derivatives are defined in a common domain. Then

(i) Derivative of sum/Difference of two functions is sum/Difference of the derivatives of the functions.

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)p$$

(ii) Derivative of product with constant.

$$\frac{d}{dx}\left(kf(x)\right) = k\frac{d}{dx}f(x)$$

Basics of Calculus

(iii) Derivative of Product of two functions is as follows

$$\frac{d}{dx}\left(f(x) \times g(x)\right) = f(x) \times \frac{d}{dx}g(x) + g(x) \times \frac{d}{dx}f(x)$$

(iv) Derivative of Division (Quotient) of two functions is as follows

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g^2(x)}$$

Example 8: $\frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 =$

(a) $1 - \frac{1}{x^2}$ (b) $1 + \frac{1}{x^2}$

(c) $1 - \frac{1}{2x}$ (d) None of these

Solution: (a) $\frac{d}{dx}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2=\frac{d}{dx}\left[x+\frac{1}{x}+2\right]=1-\frac{1}{x^2}$.

Example 9: Find the derivative of the following

(i) $(x-1)(x^2+3)$ (ii) $\frac{(2x+1)(x^2-3)}{x}$

Solution:

(i) $Let,\ y=(x-1)(x^2+3)$

$$\therefore y=x^3-x^2+3x-3$$

$$\therefore \frac{dy}{dx}=3x^2-2x+3$$

(ii) $Let,\ y=\frac{(2x+1)(x^2-3)}{x}$

$$\therefore y=\frac{2x^3+x^2-6x-3}{x}$$

$$\therefore y=2x^2+x-6-\frac{3}{x}$$

$$\therefore \frac{dy}{dx}=4x+1+\frac{3}{x^2}$$

Example 10: Find the derivative of the following

(i) $(1+2x^2)\cos x$ (ii) $2x\sin x-\left(1+x^2\right)\sin x$

Solution:

(i) $Let,\ y=(1+2x^2)\cos x$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left[(1+2x^2)\cos x\right]$$

$$\therefore \frac{dy}{dx}=(1+2x^2)\frac{d}{dx}\cos x+\cos x\frac{d}{dx}(1+2x^2)$$

$$\therefore \frac{dy}{dx}=(1+2x\ \ )\left(-\sin x\right)+\cos x\left(4x\right)$$

$$\therefore \frac{dy}{dx}=4x\cos x-2x^2\sin x-\sin x$$

(ii) *Let*, $y = 2x\sin x - \left(1+x^2\right)\sin x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[2x\sin x - \left(1+x^2\right)\sin x\right]$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(2x\sin x) - \frac{d}{dx}\left[\left(1+x^2\right)\sin x\right]$$

$$\therefore \frac{dy}{dx} = \left[2x\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(2x)\right] - \left[\left(1+x^2\right)\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}\left(1+x^2\right)\right]$$

$$\therefore \frac{dy}{dx} = \left[2x\left(\cos x\right) + \sin x\left(2\right)\right] - \left[\left(1+x^2\right)\left(\cos x\right) + \sin x\left(2x\right)\right]$$

$$\therefore \frac{dy}{dx} = \left[2x\cos x + 2\sin x\right] - \left[\cos x + x^2\cos x + 2x\sin x\right]$$

$$\therefore \frac{dy}{dx} = 2x\cos x + 2\sin x - \cos x - x^2\cos x - 2x\sin x$$

Example 11: Find $\dfrac{dy}{dx}$

(i) $y = \dfrac{x^2 - 4}{3x^2 + 5}$ (ii) $y = \dfrac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}}$

(iii) $y = \dfrac{4^x \cot x}{\sqrt{x}}$

Solution:

(i) $y = \dfrac{x^2 - 4}{3x^2 + 5}$

$$\therefore \frac{dy}{dx} = \frac{\left(3x^2+5\right)\frac{d}{dx}\left(x^2-4\right) - \left(x^2-4\right)\frac{d}{dx}\left(3x^2+5\right)}{\left(3x^2+5\right)^2}$$

$$\therefore \frac{dy}{dx} = \frac{\left(3x^2+5\right)(2x) - \left(x^2-4\right)(6x)}{\left(3x^2+5\right)^2}$$

$$\therefore \frac{dy}{dx} = \frac{6x^3 + 10x - 6x^3 + 24x}{\left(3x^2+5\right)^2}$$

$$\therefore \frac{dy}{dx} = \frac{34x}{\left(3x^2+5\right)^2}$$

(ii) $$y=\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\sqrt{a}-\sqrt{x}\right)\frac{d}{dx}\left(\sqrt{a}+\sqrt{x}\right)-\left(\sqrt{a}+\sqrt{x}\right)\frac{d}{dx}\left(\sqrt{a}-\sqrt{x}\right)}{\left(\sqrt{a}-\sqrt{x}\right)^2}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\sqrt{a}-\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}}\right)-\left(\sqrt{a}+\sqrt{x}\right)\left(\frac{-1}{2\sqrt{x}}\right)}{\left(\sqrt{a}-\sqrt{x}\right)^2}$$

$$\therefore \frac{dy}{dx}=\frac{\frac{1}{2\sqrt{x}}\left(\sqrt{a}-\sqrt{x}+\sqrt{a}+\sqrt{x}\right)}{\left(\sqrt{a}-\sqrt{x}\right)^2}$$

$$\therefore \frac{dy}{dx}=\frac{\sqrt{a}}{\sqrt{x}\left(\sqrt{a}-\sqrt{x}\right)^2}$$

(iii) $$y=\frac{4^x\cot x}{\sqrt{x}}$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\frac{4^x\cot x}{\sqrt{x}}\right)$$

$$\therefore \frac{dy}{dx}=\frac{\sqrt{x}\frac{d}{dx}\left(4^x\cot x\right)-4^x\cot x\frac{d}{dx}\sqrt{x}}{\left(\sqrt{x}\right)^2}$$

$$\therefore \frac{dy}{dx}=\frac{\sqrt{x}\left[4^x(-\cos ec^2x)+\cot x\cdot 4^x\log 4\right]-4^x\cot x\frac{1}{2\sqrt{x}}}{x}$$

$$\therefore \frac{dy}{dx}=\frac{4^x\left(2x\cot x\log 4-2x\cos ec^2x-\cot x\right)\cdot}{2x\sqrt{x}}$$

Example 12: If $f(x)=x\tan^{-1}x$, find $f'(1)=$

Solution: $f(x)=x\tan^{-1}x$

Differentiating w.r.t x, we get $f'(x)=x\frac{1}{1+x^2}+\tan^{-1}x$

Now put $x=1$, $f'(1)=\frac{1}{2}+\tan^{-1}(1)=\frac{1}{2}+\frac{\pi}{4}$

3.3 DIFFERENTIATION OF COMPOSITE FUNCTION (CHAIN RULE)

If y is a function of u and u is a function of x then the derivative of y w.r.t. x is

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

In general

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdots\cdots\frac{dz}{dx}$$

Example 13: If $y=f(x)=\sin x^2$ then differentiate y with respect to x.

Solution: $\frac{d}{dx}y=\frac{d}{dx}sinx^2=cosx^2\frac{d}{dx}x^2=2x\,cosx^2$

Example 14: Find $\frac{d}{dx}(\log\tan x)=$

Solution: $\frac{d}{dx}(\log\tan x)=\frac{1}{\tan x}\sec^2 x=\frac{\cos x}{\cos^2 x\sin x}$

$$=\frac{2}{2}\frac{1}{\cos x\sin x}=2\operatorname{cosec}2x$$

Example 15: $\frac{d}{dx}\tan^{-1}(\sec x+\tan x)=$

Solution: $\frac{d}{dx}\tan^{-1}(\sec x+\tan x)=\frac{d}{dx}\tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)$

$$=\frac{d}{dx}\tan^{-1}\left(\frac{\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}\right)=\frac{d}{dx}\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)=\frac{d}{dx}\left(\frac{\pi}{4}+\frac{x}{2}\right)=\frac{1}{2}$$

Example 16: Differentiate $\cos ec(2x^2+3)$ with respect to x

Solution: $Let,\ y=\cos ec(2x^2+3)$

$$\therefore\frac{dy}{dx}=\frac{d}{dx}\left[\cos ec(2x^2+3)\right]$$

$$\therefore\frac{dy}{dx}=-\cos ec(2x^2+3)\cot(2x^2+3)\cdot\frac{d}{dx}(2x^2+3)$$

$$\therefore\frac{dy}{dx}=-4x\cos ec(2x^2+3)\cot(2x^2+3)$$

Example 17: Differentiate the following functions with respect to x

(i) $\sqrt{\dfrac{1-\cos x}{1+\cos x}}$ (ii) $\tan^3 x+\dfrac{1}{3}\tan x+\dfrac{x}{3}$

Solution:

(i) $Let,\ y=\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left[\sqrt{\frac{1-\cos x}{1+\cos x}}\right]=\frac{1}{2\sqrt{\dfrac{1-\cos x}{1+\cos x}}}\cdot\frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$\therefore \frac{dy}{dx}=\frac{1}{2\sqrt{\dfrac{1-\cos x}{1+\cos x}}}\cdot\left[\frac{(1+\cos x)\cdot\dfrac{d}{dx}(1-\cos x)-(1-\cos x)\cdot\dfrac{d}{dx}(1+\cos x)}{(1+\cos x)^2}\right]$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{1+\cos x}{1-\cos x}}\cdot\left[\frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}\right]$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{1+\cos x}{1-\cos x}}\cdot\left[\frac{2\sin x}{(1+\cos x)^2}\right]=\frac{1}{1+\cos x}$$

Note that, in this example if you simplify y using the trigonometric identities then alternatively it can be solved easily, this is the beauty of Mathematics. You arrive at

$$\frac{dy}{dx}=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)$$

Which further simplifies $\dfrac{1}{2}\sec^2\left(\dfrac{x}{2}\right)=\dfrac{1}{1+\cos x}$

(ii) $Let,\ y=\tan^3 x+\dfrac{1}{3}\tan x+\dfrac{x}{3}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\tan^3 x+\frac{1}{3}\tan x+\frac{x}{3}\right)$$

$$\therefore \frac{dy}{dx}=3\tan^2 x\frac{d}{dx}(\tan x)+\frac{1}{3}\sec^2 x+\frac{1}{3}$$

$$\therefore \frac{dy}{dx}=3\tan^2 x\sec^2 x+\frac{1}{3}\sec^2 x+\frac{1}{3}$$

3.4 DIFFERENTIATION OF TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

Example 18: Find $\frac{dy}{dx}$

(i) $y=\sqrt{1+\sin 2x}+\frac{\sqrt{1+\cos 2x}}{1-\cos 2x}$ (ii) $y=\sec^2 x+\tan^2 x$

(iii) $y=\frac{1-\sin x}{\cos x}$

Solution:

(i)
$$y=\sqrt{1+\sin 2x}+\frac{\sqrt{1+\cos 2x}}{1-\cos 2x}$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{1+\sin 2x}+\frac{\sqrt{1+\cos 2x}}{1-\cos 2x}\right)=\frac{d}{dx}\left(\sqrt{1+\sin 2x}\right)+\frac{d}{dx}\left(\frac{\sqrt{1+\cos 2x}}{1-\cos 2x}\right)$$

$$=\frac{1}{2\sqrt{1+\sin 2x}}\frac{d}{dx}(1+\sin 2x)+\frac{d}{dx}\left(\frac{\sqrt{1+2\cos^2 x-1}}{1-1+2\sin^2 x}\right)$$

$$=\frac{1}{2\sqrt{1+\sin 2x}}\left[\frac{d}{dx}(1)+\frac{d}{dx}\sin 2x\right]+\frac{d}{dx}\left(-\frac{1}{\sqrt{2}}\cot x\cdot\cos ecx\right)$$

$$=\frac{1}{2\sqrt{1+\sin 2x}}[2\cos 2x]-\frac{1}{\sqrt{2}}\left(\cot x\frac{d}{dx}(\cos ecx)+\cos ecx\frac{d}{dx}(\cot x)\right)$$

$$=\frac{\cos 2x}{\sqrt{1+\sin 2x}}-\frac{1}{\sqrt{2}}\left[\cot x(-\cos ecx\cot x)+\cos ecx(-\cos ec^2 x)\right]$$

$$=\frac{\cos 2x}{\sqrt{1+\sin 2x}}+\frac{\cos ecx}{\sqrt{2}}\left[\cot^2 x+\cos ec^2 x\right]$$

(ii) $y=\sec^2 x+\tan^2 x$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\sec^2 x+\tan^2 x\right)=\frac{d}{dx}\left(\sec^2 x\right)+\frac{d}{dx}\left(\tan^2 x\right)$$

$$=2\sec x\cdot\sec x\tan x+2\tan x\cdot\sec^2 x=4\tan x\cdot\sec^2 x$$

or

$$y=\sec^2 x+\tan^2 x=\sec^2 x-1+\sec^2 x=2\sec^2 x-1$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\sec^2 x+\tan^2 x\right)=\frac{d}{dx}\left(2\sec^2 x-1\right)=4\sec x\cdot\sec x\tan x$$

$$=4\tan x\cdot\sec^2 x$$

(iii) $y=\dfrac{1-\sin x}{\cos x}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\frac{1-\sin x}{\cos x}\right)=\frac{\cos x\frac{d}{dx}(1-\sin x)-(1-\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$=\frac{\cos x(-\cos x)-(1-\sin x)(-\sin x)}{\cos^2 x}=\frac{-\cos^2 x+\sin x-\sin^2 x}{\cos^2 x}$$

$$=\frac{\sin x-1}{\cos^2 x}$$

Example 19: Find $\dfrac{dy}{dx}$

(i) $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ **(ii)** $y=\tan^{-1}\left(\sqrt{\dfrac{1-x}{1+x}}\right)$

(iii) $y=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

Solution:

(i) $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$,

$$let,\ x=\sin\theta \qquad \therefore \theta=\sin^{-1}x$$

$$and\ \ 2x\sqrt{1-x^2}=2\sin\theta\sqrt{1-\sin^2\theta}=2\sin\theta\cos\theta=\sin 2\theta$$

$$\therefore y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)=\sin^{-1}(\sin 2\theta)=2\theta=2\sin^{-1}x$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(2\sin^{-1}x\right)=2\frac{1}{\sqrt{1-x^2}}=\frac{2}{\sqrt{1-x^2}}$$

(ii) $y=\tan^{-1}\left(\sqrt{\dfrac{1-x}{1+x}}\right)$,

$$let,\ x=\cos\theta \qquad \therefore \theta=\cos^{-1}x$$

$$and\ \ \sqrt{\frac{1-x}{1+x}}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\tan\frac{\theta}{2}$$

$$\therefore y=\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}=\frac{1}{2}\cos^{-1}x$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{2}\cos^{-1}x\right)=\frac{1}{2}\frac{-1}{\sqrt{1-x^2}}=\frac{-1}{2\sqrt{1-x^2}}$$

(iii) $y=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$,

$$let,\ x=\tan\theta \qquad \therefore \theta=\tan^{-1}x$$

$$and\ \frac{2x}{1+x^2}=\frac{2\tan\theta}{1+\tan^2\theta}=\tan 2\theta$$

$$\therefore y=\tan^{-1}\left(\frac{2x}{1+x^2}\right)=\tan^{-1}(\tan 2\theta)=2\theta=2\tan^{-1}x$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(2\tan^{-1}x\right)=2\frac{1}{1+x^2}=\frac{2}{1+x^2}$$

Example 20: Find $\dfrac{dy}{dx}$

(i) $y=\sin^{-1}\left(3x-4x^3\right)$ (ii) $y=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(iii) $y=\tan^{-1}\left(\dfrac{1-\sqrt{1-x^2}}{x}\right)$

Solution:

(i) $y=\sin^{-1}\left(3x-4x^3\right)$,

$$let,\ x=\sin\theta \qquad \therefore \theta=\sin^{-1}x$$

$$and\ 3x-4x^3=3\sin\theta-4\sin^3\theta=\sin 3\theta$$

$$\therefore y=\sin^{-1}\left(3x-4x^3\right)=\sin^{-1}(\sin 3\theta)=3\theta=3\sin^{-1}x$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(3\sin^{-1}x\right)=3\frac{1}{\sqrt{1-x^2}}=\frac{3}{\sqrt{1-x^2}}$$

(ii) $y=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$,

$$let,\ x=\tan\theta \qquad \therefore \theta=\tan^{-1}x$$

$$and\ \frac{1-x^2}{1+x^2}=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta$$

$$\therefore y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\cos^{-1}(\cos 2\theta)=2\theta=2\tan^{-1}x$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(2\tan^{-1}x\right)=2\frac{1}{1+x^2}=\frac{2}{1+x^2}$$

(iii) $y=\tan^{-1}\left(\dfrac{1-\sqrt{1-x^2}}{x}\right)$,

$$let,\ x=\sin\theta \qquad \therefore \theta=\sin^{-1}x$$

$$and\ \frac{1-\sqrt{1-x^2}}{x}=\frac{1-\cos\theta}{\sin\theta}=\tan\frac{\theta}{2}$$

$$\therefore y=\tan^{-1}\left(\frac{1-\sqrt{1-x^2}}{x}\right)=\tan^{-1}(\tan\frac{\theta}{2})=\frac{\theta}{2}=\frac{1}{2}\sin^{-1}x$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{2}\sin^{-1}x\right)=\frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}=\frac{1}{2\sqrt{1-x^2}}$$

3.5 DIFFERENTIATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

Example 21: Find $\dfrac{dy}{dx}$

(i) $y=x^{\sqrt{x}}$ (ii) $y=\left(\sin x\right)^x+x^{\sin x}$

(iii) $y=\log_{\cos x}\sin x$

Solution:

(i) $y=x^{\sqrt{x}}$,

taking log *in both the side,* $\log y=\log\left(x^{\sqrt{x}}\right)$

$$\therefore \log y=\sqrt{x}\log x$$

$$\therefore \frac{1}{y}\frac{dy}{dx}=\sqrt{x}\frac{d}{dx}\left(\log x\right)+\log x\frac{d}{dx}(\sqrt{x})$$

$$\therefore \frac{1}{y}\frac{dy}{dx}=\sqrt{x}\cdot\frac{1}{x}+\log x\cdot\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{x}}(2+\log x)$$

$$\therefore \frac{dy}{dx}=y\left[\frac{1}{2\sqrt{x}}(2+\log x)\right]=x^{\sqrt{x}}\left[\frac{1}{2\sqrt{x}}(2+\log x)\right]$$

(ii) $y = (\sin x)^x + x^{\sin x}$,

$let\ us\ assume\ that,\ u = (\sin x)^x\ and\ v = x^{\sin x}$

$\therefore \log u = \log(\sin x)^x = x \log(\sin x)$

$\therefore \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\left[\log(\sin x)\right] + \log(\sin x)\frac{d}{dx}(x)$

$\therefore \frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{\sin x} \cdot \cos x + \log(\sin x) \cdot 1 = x \cot x + \log(\sin x)$

$\therefore \frac{du}{dx} = u\left[x \cot x + \log(\sin x)\right] = (\sin x)^x \left[x \cot x + \log(\sin x)\right]$

$and\ v = x^{\sin x}$

$\therefore \log v = \log(x)^{\sin x} = \sin x \log(x)$

$\therefore \frac{1}{v}\frac{dv}{dx} = \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x)$

$\therefore \frac{1}{v}\frac{dv}{dx} = \sin x \cdot \frac{1}{x} + \log x \cdot \cos x = \frac{\sin x}{x} + \cos x \log x$

$\therefore \frac{dv}{dx} = v\left[\frac{\sin x}{x} + \cos x \log x\right] = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = (\sin x)^x \left[x \cot x + \log(\sin x)\right] + x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$

(iii) $y = \log_{\cos x} \sin x$

$here,\ y = \log_{\cos x} \sin x \qquad \therefore y = \frac{\log_e \sin x}{\log_e \cos x}$

$now\ let,\ u = \log_e \sin x \qquad and \qquad v = \log_e \cos x$

$\therefore \frac{du}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x\ and \qquad \frac{dv}{dx} = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x$

$\therefore y = \frac{u}{v}\quad and \qquad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\therefore \frac{dy}{dx} = \frac{(\log_e \cos x)(\cot x) - (\log_e \sin x)(-\tan x)}{(\log_e \cos x)^2}$

$\therefore \frac{dy}{dx} = \frac{\cot x \log_e \cos x + \tan x \log_e \sin x}{(\log_e \cos x)^2}$

Example 22: Find $\frac{dy}{dx}$

$$y = x^{x^x}$$

Solution:

Here $y = x^{x^x}$

$$\therefore \log y = \log x^{x^x} = x^x \log x$$

$$now,\ let\ u = x^x \qquad \therefore \frac{d}{dx}x^x = \frac{du}{dx}$$

$$\log u = \log x^x = x \log x$$

$$\therefore \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x)$$

$$\therefore \frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{x} + \log x = 1 + \log x$$

$$\therefore \frac{du}{dx} = u(1+\log x) = x^x(1+\log x)$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = x^x \frac{d}{dx}\log x + \log x \frac{d}{dx}x^x$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = x^x \cdot \frac{1}{x} + x^x \log x(1+\log x)$$

$$\therefore \frac{dy}{dx} = yx^x\left[\frac{1}{x} + \log x(1+\log x)\right] = x^{x^x} \cdot x^x\left[\frac{1}{x} + \log x(1+\log x)\right]$$

APPLICATIONS (REAL LIFE / INDUSTRIAL)

Civil Engineering, Topography

Example-1: Let p(x) be the height of a road, or the elevation of a foothill, as we move along a horizontal distance x. The derivative p'(x) with respect to distance is the gradient of the road or foothill.

Newton's Method

Example-2: Useful for those equations which cannot be solved using algebra**.**

Curvilinear Motion

Example-3: Useful for finding velocity and acceleration of a body moving in a curve.

Curve Sketching Using Differentiation

Example-4: To model the behavior of variables.

Applied Maximum and Minimum Problems

Example-5: Finding the condition for a maximum (or minimum) to occur, in case when we come across the problems related to cost reduction or to increase profit.

Mechanical engineering

Example-6: Suppose that the total amount of energy produced by an engine is R(t) at time t. The derivative R'(t) of energy with respect to time is the power of the engine.

Biology

Example-7: Biologists use differential calculus to regulate the exact rate of growth in a bacterial culture.

Economics

Example-8: In macroeconomics, the rate of change of the GDP of an economy with respect to time is known as the economic growth rate. It is frequently used by economists as a measure of growth.

Computer Algebra Systems

Example-9: Maple and Mathematica, can be used to compute derivatives and is very helpful for life science students to dominant the mechanics of differentiation.

Determining Frictional Forces

Example-10: Mechanical engineer when compute the surface area of complex objects to determine frictional forces use Calculus.

Astronomy

Example-11: Based on the laws of planetary motion, Astronomers use calculus to study the different motions of planets.

Neurology

Example-12: Neurologist use differential calculus to compute the change of voltage in a neuron with respect to time.

Epidemiology

Example-13: To determine how far and how fast a disease can spread, Epidemiologists, employ calculus to study the spread of infectious diseases.

Optimization Problem

Example-14: Using differentiation stationary points of functions can be calculated, in order to sketch their graphs. Calculating stationary points helps us to solve the problems that require some variable to be maximized or minimized. Using this concept one can determine the speed of the car which uses the least amount of fuel.

UNIT SUMMARY

In this unit the first topic is devoted to present, Differentiation by definition using first principle, the Concept of average rate of change, instantaneous rate of change has been discussed. Second topic deals with Rules for differentiation with Linearity rules, Product rule, Quotient rule. These general rules are very useful for reducing differentiation problems to the basic formulas. Differentiation of function of a function i,e Chain rule is vital in many applications of mathematics to use computer algebra systems to compute derivatives. Third topic starts with Differentiation of trigonometric and inverse trigonometric functions further deals with Logarithmic differentiation and Exponential function. Each topic is followed by worked examples along with increasing level of difficulties as per revised Bloom's Taxonomy, similar pattern is adopted while providing the exercise for practice. New examples designed to reinforce the main concepts and applications of derivatives, so that students have some familiarity with the derivative as an analytical tool. Some open questions are also advised these Verbal questions will help for assessing conceptual understanding of key terms and concepts. On the other hand, Algebraic problems will help students to apply algebraic manipulations, problems related with derivative as product of graph assess students' ability to interpret derivative from graphical approach. Numeric problems require the student perform calculations or computations. Real-World Applications present realistic problem scenarios.

EXERCISES

Multiple Choice Questions

1. $\frac{d}{dx}\log(\log x) =$

(a) $\frac{x}{\log x}$ (b) $\frac{\log x}{x}$

(c) $(x\log x)^{-1}$ (d) None of these

2. $\frac{d}{dx}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 =$

(a) $1-\frac{1}{x^2}$ (b) $1+\frac{1}{x^2}$

(c) $1-\frac{1}{2x}$ (d) None of these

3. If $y = x+\frac{1}{x}$, then

(a) $x^2\frac{dy}{dx}+xy=0$ (b) $x^2\frac{dy}{dx}+xy+2=0$

(c) $x^2\frac{dy}{dx}-xy+2x^2=0$ (d) None of these

4. $\frac{d}{dx}\left(\frac{1}{x^4 \sec x}\right)=$

(a) $\frac{x\sin x+4\cos x}{x^5}$

(b) $\frac{-(x\sin x+4\cos x)}{x^5}$

(c) $\frac{4\cos x-x\sin x}{x^5}$

(d) None of these

5. If $y=x\sin x$, then

(a) $\frac{1}{y}\frac{dy}{dx}=\frac{1}{x}+\cot x$

(b) $\frac{dy}{dx}=\frac{1}{x}+\cot x$

(c) $\frac{1}{y}\frac{dy}{dx}=\frac{1}{x}-\cot x$

(d) None of these

6. $\frac{d}{dx}(x^2 e^x \sin x)=$

(a) $x e^x(2\sin x+x\sin x+x\cos x)$

(b) $x e^x(2\sin x+x\sin x-\cos x)$

(c) $x e^x(2\sin x+x\sin x+\cos x)$

(d) None of these

7. $\frac{d}{dx}\left[\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)\right]=$

(a) $-\frac{1}{2}$

(b) $\frac{1}{2}$

(c) -1

(d) 1

8. $\frac{d}{dx}[\cos(1-x^2)^2]=$

(a) $-2x(1-x^2)\sin(1-x^2)^2$

(b) $-4x(1-x^2)\sin(1-x^2)^2$

(c) $4x(1-x^2)\sin(1-x^2)^2$

(d) $-2(1-x^2)\sin(1-x^2)^2$

9. $\frac{d}{dx}\left(x^2 \sin\frac{1}{x}\right)=$

(a) $\cos\left(\frac{1}{x}\right)+2x\sin\left(\frac{1}{x}\right)$

(b) $2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)$

(c) $\cos\left(\frac{1}{x}\right)-2x\sin\left(\frac{1}{x}\right)$ (d) None of these

10. If $y=\cos(\sin x^2)$, then at $x=\sqrt{\frac{\pi}{2}}, \frac{dy}{dx}=$

(a) – 2 (b) 2

(c) $-2\sqrt{\frac{\pi}{2}}$ (d) 0

11. The differential coefficient of $a^x+\log x.\sin x$ is

(a) $a^x \log_e a+\frac{\sin x}{x}+\log x.\cos x$ (b) $a^x+\frac{\sin x}{x}+\cos x.\log x$

(c) $a \ \log a+\frac{\cos}{}+\sin x.\log x.$ (d) None of these

12. $\frac{d}{dx}\tan^{-1}\left(\frac{ax-b}{bx+a}\right)=$

(a) $\frac{1}{1+x^2}-\frac{a^2}{a^2+b^2}$ (b) $\frac{-1}{1+x^2}-\frac{a^2}{a^2+b^2}$

(c) $\frac{1}{1+x^2}+\frac{a^2}{a^2+b^2}$ (d) None of these

13. $\frac{d}{dx}\left(\tan^{-1}\sqrt{\frac{1+\cos\frac{x}{2}}{1-\cos\frac{x}{2}}}\right)$ is equal to

(a) $-\frac{1}{4}$ (b) $\frac{1}{2}$

(c) $-\frac{1}{2}$ (d) $\frac{1}{4}$

14. $\frac{d}{dx}\sqrt{\frac{1-\sin 2x}{1+\sin 2x}}=$

(a) $\sec^2 x$ (b) $-\sec^2\left(\frac{\pi}{4}-x\right)$

(c) $\sec^2\left(\frac{\pi}{4}+x\right)$ (d) $\sec^2\left(\frac{\pi}{4}-x\right)$

15. If $y=\sqrt{(1-x)(1+x)}$, then

(a) $(1-x^2)\frac{dy}{dx}-xy=0$

(b) $(1-x^2)\frac{dy}{dx}+xy=0$

(c) $(1-x^2)\frac{dy}{dx}-2xy=0$

(d) $(1-x^2)\frac{dy}{dx}+2xy=0$

16. $\frac{d}{dx}\left(\frac{\cot^2 x-1}{\cot^2 x+1}\right)=$

(a) $-\sin 2x$

(b) $2\sin 2x$

(c) $2\cos 2x$

(d) $-2\sin 2x$

17. $\frac{d}{dx}[\sin^n x\cos nx]=$

(a) $n\sin^{n-1} x\cos(n+1)x$

(b) $n\sin^{n-1} x\cos nx$

(c) $n\sin^{n-1} x\cos(n-1)x$

(d) $n\sin^{n-1} x\sin(n+1)x$

18. If $f(x)=\log_x(\log x)$, then $f'(x)$ at $x=e$ is

(a) e

(b) $\frac{1}{e}$

(c) 1

(d) None of these

19. If $y=\log\left(\frac{1+x}{1-x}\right)^{1/4}-\frac{1}{2}\tan^{-1}x$, then $\frac{dy}{dx}=$

(a) $\frac{x^2}{1-x^4}$

(b) $\frac{2x^2}{1-x^4}$

(c) $\frac{x^2}{2\,(1-x^4)}$

(d) None of these

20. If $y=\log\frac{1+\sqrt{x}}{1-\sqrt{x}}$, then $\frac{dy}{dx}=$

(a) $\frac{\sqrt{x}}{1-x}$

(b) $\frac{1}{\sqrt{x}(1-x)}$

(c) $\frac{\sqrt{x}}{1+x}$

(d) $\frac{1}{\sqrt{x}(1+x)}$

21. $\frac{d}{dx}e^{x+3\log x}=$

(a) $e^x.x^2(x+3)$ (b) $e^x.x(x+3)$

(c) $e^x+\frac{3}{x}$ (d) None of these

22. $\frac{d}{dx}\sqrt{\frac{1+\cos 2x}{1-\cos 2x}}=$

(a) $\sec^2 x$ (b) $-\text{cosec}^2 x$

(c) $2\sec^2\frac{x}{2}$ (d) $-2\text{cosec}^2\frac{x}{2}$

23. $\frac{d}{dx}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=$

(a) $\text{cosec}\, x$ (b) $-\text{cosec}\, x$

(c) $\sec x$ (d) $-\sec x$

24. $\frac{d}{dx}\log(\sqrt{x-a}+\sqrt{x-b})=$

(a) $\frac{1}{2[\sqrt{(x-a)}+\sqrt{(x-b)}]}$ (b) $\frac{1}{2\sqrt{(x-a)(x-b)}}$

(c) $\frac{1}{\sqrt{(x-a)(x-b)}}$ (d) None of these

25. If $y=\sin[\cos(\sin x)]$, then $dy/dx=$

(a) $-\cos[\cos(\sin x)]\sin(\cos x).\cos x$

(b) $-\cos[\cos(\sin x)]\sin(\sin x).\cos x$

(c) $\cos[\cos(\sin x)]\sin(\cos x).\cos x$

(d) $\cos[\cos(\sin x)]\sin(\sin x).\cos x$

26. If $y=\sin^{-1}\sqrt{\sin x}$, then $\frac{dy}{dx}$ equals to

(a) $\frac{2\sqrt{\sin x}}{\sqrt{1+\sin x}}$ (b) $\frac{\sqrt{\sin x}}{\sqrt{1-\sin x}}$

(c) $\frac{1}{2}\sqrt{1+\cos ec\, x}$ (d) $\frac{1}{2}\sqrt{1-\cos ec\, x}$

27. If y = log (cosec x – cot x), then $\frac{dy}{dx}$ equals to

(a) cosec x + cot x (b) cot x

(c) sec x + tan x (d) cosec x

28. If y = log (sinx), then $\frac{dy}{dx}$ equals to

(a) cot x (b) tan x

(c) sec x (d) cosec x

29. If y = sin(logx) then $\frac{dy}{dx}$ equals to

(a) $\cos(\log x)$ (b) $\frac{1}{sinx}$

(c) $\frac{\cos(logx)}{x}$ (d) cosec (log$x$)

30. If y = $3\sqrt[3]{cosx}$ then $\frac{dy}{dx}$ equals to -

(a) $3\sqrt[3]{sinx}$ (b) $\frac{1}{2\sqrt[3]{cosx}}$

(c) $_{-}3\sqrt[3]{sinx}$ (d) $\frac{-sinx}{\sqrt[3]{cos^2x}}$

Answers of Multiple-Choice Questions									
1.	c	**2.**	a	**3.**	c	**4.**	b	**5.**	a
6.	a	**7.**	a	**8.**	c	**9.**	b	**10.**	d
11.	a	**12.**	d	**13.**	a	**14.**	b	**15.**	b
16.	d	**17.**	a	**18.**	b	**19.**	a	**20.**	b
21.	a	**22.**	b	**23.**	c	**24.**	b	**25.**	b
26.	c	**27.**	d	**28.**	a	**29.**	c	**30.**	d

Short and Long Answer Type Questions

1. Find the derivatives of the following functions using the definition.

(i) $y = x^2 + 2x - 3$ (ii) $y = \cos 2x$

(iii) $y = \log(x+1)$ (iv) $y = \frac{1}{2x+3}$

(v) $y = e^{3x+2}$ (vi) $y = \tan x$

2. Find the derivatives of the following functions.

(i) $y = \sqrt{x} - \dfrac{1}{\sqrt{x}}$ (ii) $y = \dfrac{4x^3 - 3x + 7}{x}$

(iii) $y = \dfrac{\left(x^2+1\right)(x-1)}{x\sqrt{x}}$ (iv) $y = 2x^3 - \dfrac{1}{\sqrt{x}} + 2^x$

(v) $y = \left(3x^2 - 2x + 5\right)^2$ (vi) $y = \log_e x^3 + 3e^x - 12$

3. Find the derivatives of the following functions.

(i) $y = (2 + 3\sin x)(3 + 2\cos x)$ (ii) $y = x\tan x(x^2+1)$

(iii) $y = \cos ecx \cot x$ (iv) $y = x\sin^{-1} x$

(v) $y = e^x \tan^{-1} x$ (vi) $y = e^x x^2 2^x$

4. Find the derivatives of the following functions.

(i) $y = \dfrac{px^2 + qx + r}{ax^2 + bx + c}$ (ii) $y = \dfrac{\sin x}{1 + \cos x}$

(iii) $y = \dfrac{\tan x}{\sec x + \tan x}$ (iv) $y = \dfrac{x^2 - a^2}{x^2 + a^2}$

(v) $y = \dfrac{\sin x}{x}$ (vi) $y = \dfrac{e^x \tan x}{x}$

5. Differentiate $y = \dfrac{x^2 - 3x + 7}{x^2 - 5}$ w.r.t. x and find $\left(\dfrac{dy}{dx}\right)_{x=1}$

6. If $y = \dfrac{x}{1+x}, x \neq -1$, prove that $x^2 \dfrac{dy}{dx} = y^2$.

7. Differentiate the following functions w.r.t. x.

(i) $y = \tan^3 x + \tan x^3$ (ii) $y = \sqrt{\dfrac{1-x}{1+x}}$

(iii) $y = \log\left(x + \sqrt{x^2 + a^2}\right)$

8. Differentiate the following functions w.r.t. x.

(i) $y = e^{2x} \sin 3x$ (ii) $y = 2^x \log \cos x$

(iii) $y = \log\left[\log(\sin 2x)\right]$ (iv) $y = e^{m\tan^{-1}x}$

(v) $y = \left(3x^3 + 2x^2 - x - 11\right)^5$ (vi) $y = \sin\left[\log(\cos 2x)\right]$

9. Find the derivatives of the following functions.

(i) $y = (\sin x)^{\tan x}$ (ii) $(\cos x)^y = (\sin y)^x$

(iii) $y = \dfrac{x+1}{(x-3)(x+2)}$ (iv) $y = (x)^{\sqrt{x}} + \left(\sqrt{x}\right)^x$

(v) $y = x^{x^{\sqrt{x}}}$ (vi) $x^y = e^{x+y}$

10. Find the derivatives of the following functions w.r.t. x.

(i) $y = \cos^{-1}\left(\dfrac{x}{x+1}\right)$ (ii) $y = \tan^{-1}\left(\dfrac{\sin 3\theta - \sin\theta}{\cos 3\theta + \cos\theta}\right)$

(iii) $y = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$ (iv) $y = \tan^{-1}\left[\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right]$

(v) $y = \cos^{-1}\left[\sin\left(\dfrac{5\pi}{2} + \dfrac{x^2+1}{x^2-1}\right)\right]$ (vi) $y = \cos^{-1}\left(2x^2 - 1\right)$

11. Differentiate the following functions w.r.t. x.

(i) $x = at^2, y = 2at \quad t \in R$

(ii) $x = a\sec\theta, y = b\tan\theta \text{ where } \theta \neq (2k+1)\dfrac{\pi}{2}, k \in Z$

(iii) $x = 2\cos t - \cos 2t, y = 2\sin t - \sin 2t,\ t \in R$

12. Differentiate the following functions w.r.t. x.

(i) $x^3 + y^3 = 3xy$ (ii) $y = \sin(x+y)$

(iii) $x\sin y + y\sin x = 11$

Answers of Short and Long Answer Type Questions

1. (i) $2x+2$ (ii) $-2\sin 2x$ (iii) $\dfrac{1}{x+1}$

(iv) $\dfrac{-2}{(2x+3)^2}$ (v) $3e^{3x+2}$ (vi) $\sec^2 x$

Answers of Short and Long Answer Type Questions

2. (i) $\frac{1}{2}\left(x^{\frac{-1}{2}}+x^{\frac{-3}{2}}\right)$ (ii) $8x-\frac{7}{x^2}$ (iii) $\frac{1}{2}\left(3x^{\frac{1}{2}}-x^{\frac{-1}{2}}-x^{\frac{-3}{2}}+3x^{\frac{-5}{2}}\right)$

(iv) $6x^2+\frac{1}{2}x^{\frac{-3}{2}}+2^x\log 2$ (v) $2\left(18x^3-18x^2+34x-10\right)$ (vi) $3\left(\frac{1}{x}+e^x\right)$

3. (i) $9\cos x-4\sin x+6\cos 2x$ (ii) $(x^3+x)\sec^2 x+(3x^2+1)\tan x$

(iii) $\frac{-(1+\cos^2 x)}{\sin^3 x}$ (iv) $\frac{x}{\sqrt{1-x^2}}+\sin^{-1}x$ (v) $e^x\left(\frac{1}{1+x^2}+\tan^{-1}x\right)$

(vi) $xe^x 2^x\left(x\log 2+x+2\right)$

4. (i) $\frac{(pb-qa)x^2+2(pc-ra)x+(qc-rb)}{\left(ax^2+bx+c\right)^2}$ (ii) $\frac{1}{1+\cos x}$ (iii) $\frac{\sec x}{\left(\sec x+\tan x\right)^2}$

(iv) $\frac{4xa^2}{\left(x^2+a^2\right)^2}$ (iv) $\frac{x\cos x-\sin x}{x^2}$ (vi) $\frac{e^x}{x^2}\left[x\sec^2 x+\left(x-1\right)\tan x\right]$

5. $\frac{3x^2-24x+15}{\left(x^2-5\right)^2}$ $,-\frac{3}{8}$

7. (i) $3\left[\tan^2 x\sec^2 x+x^2\sec^2\left(x^3\right)\right]$ (ii) $\frac{-1}{\left(\sqrt{1-x^2}\right)\left(1+x\right)}$ (iii) $\frac{1}{\sqrt{x^2+a^2}}$

8. (i) $e^{2x}(3\cos 3x+2\sin 3x)$ (ii) $2^x(\log 2\cdot\log\cos x-\tan x)$ (iii) $\frac{2\cot 2x}{\log(\sin 2x)}$

(iv) $\frac{m}{1+x^2}e^{m\tan^{-1}x}$ (v) $5(9x^2+4x-1)(3x^3+2x^2-x-11)^4$

(vi) $-2\tan 2x\cdot\cos\left[\log\left(\cos 2x\right)\right]$

9. (i) $(\sin x)^{\tan x}\left[1+\sec^2 x\log\sin x\right]$ (ii) $\frac{\log(\sin y)+y\tan x}{\log(\cos x)-x\cot y}$ (iii) $\frac{-\left(x^2+2x+5\right)}{(x-3)^2(x+2)^2}$

Answers of Short and Long Answer Type Questions

(iv) $\dfrac{x^{\sqrt{x}}}{2\sqrt{x}}(2+\log x)+\dfrac{\left(\sqrt{x}\right)^{x}}{2}\left(1+2\log\sqrt{x}\right)$ (iv) $x^{x^{\sqrt{x}}}x^{\sqrt{x}}\left(\dfrac{1}{x}+\dfrac{\log x(2+\log x)}{2\sqrt{x}}\right)$

(vi) $\dfrac{x-y}{x(\log x-1)}$

10. (i) $\dfrac{1}{(x+1)\sqrt{2x+1}}$ (ii) 0 (iii) $\dfrac{3}{1+x^2}$ (iv) $\dfrac{1}{2}$

(v) $\dfrac{\quad}{(\quad)}$ (vi) $\dfrac{2}{\sqrt{1-x^2}}$

11. (i) $\dfrac{1}{t}$ (ii) $\dfrac{b}{a}\cos ec\theta$ (iii) $\tan\left(\dfrac{3t}{2}\right)$

12. (i) $\dfrac{y-x^2}{y^2-x}$ (ii) $\dfrac{\cos(x+y)}{1-\cos(x+y)}$ (iii) $-\dfrac{\sin y+y\cos x}{\sin x+x\cos y}$

KNOW MORE

- Development of Differential Calculus.
- Why Study Differentiation?
- How calculus is used in our daily lives.
- Limits and Differentiation
- The Slope of a Tangent to a Curve.
- Derivative as an Instantaneous Rate of Change
- Derivatives of Polynomials
- Higher Derivatives
- Partial Derivatives
- Why mathematical thinking is valuable in daily life
- Shift to Online Teaching.
- The simplest way to learn Differentiation.
- Why was Differential Calculus invented?
- Learning of Differential Calculus intuitively.
- Making of Differentiation less complicated.
- Online Education Tools for Teachers.
- Teaching Critical Thinking
- STEM Education.

Mini Project

i. Use Geo Gebra to plot the graph of functions and its derivatives viewing Engineering applications.
ii. Collect set of problems based on applications of derivative for real world problems.

Inquisitiveness and Curiosity Topics

i. In a company, how profit can be maximized or how can we minimize the costs?
ii. How do we calculate the minimum amount of material to make a particular object?
iii. Why a car has skidded while rounding a turn?
iv. What happen with Non-polynomial Functions with Multiple Roots?
v. How to choose the best stocks?
vi. How do we determine whether the given equation is a linear equation or a nonlinear equation?
vii. Can a function which is not defined at a point, be differentiable at that point?
viii. The sum of two positive numbers is 40. One of the numbers is multiplied by the square of the other. In order to maximize the product, what is the corresponding number?
ix. How do you measure your position precisely at every instant?
x. Can any statement about tangents of a graph be recited as statements about derivatives?

Apart from the above questions, Calculus is used to solve the area of complicated and irregular shapes, estimating survey data, the safety of vehicles, occupational forecasting, credit card payment records, etc.

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4 Complex Numbers and Partial Fraction

UNIT SPECIFICS

This unit elaborately discusses the following topics:

- Definition of real and imaginary parts of a Complex number;
- Polar and Cartesian form;
- Representation of a complex number and its conversion from one form to other;
- Conjugate of a complex number, modulus and amplitude of a complex number;
- Addition, Subtraction, Multiplication and Division of a complex number;
- De-Movire's theorem, its applications;
- Definition of polynomial fraction proper & improper fractions;
- Definition of partial fractions;
- To resolve proper fraction into partial fraction under various case;
- To resolve improper fraction into partial fraction.

The applications-based problems are discussed for generating further curiosity and creativity as well as improving problem solving capacity.

Besides giving a large number of multiple-choice questions as well as questions of short and long answer types marked in two categories following lower and higher order of Bloom's taxonomy, assignments through a number of numerical problems, a list of references and suggested readings are given in the unit so that one can go through them for practice.

Based on the content, there is "Know More" section added. This section has been thoughtfully planned so that the supplementary information provided in this part becomes beneficial for the users of the book. This section mainly highlights further teaching and learning related to some interesting facts about why study Complex numbers? how Complex numbers and Partial fractions are used in our daily lives, Realize the need to expand the set of real numbers, why mathematical thinking is valuable in daily life, shift to Online Teaching. The simplest way to learn Complex numbers and its algebra, Learning and importance of Partial fraction intuitively, Online Education Tools for Teachers.

On the other hand, suggested Micro projects and brain storming questions create inquisitiveness and curiosity for the topics included in the unit.

RATIONALE

Complex numbers, is one of the most elegant and interesting topics in mathematics. Complex numbers, their algebra and geometry has always been an important tool to crack thousands of the problems based on Pure and Applied Mathematics. In fact, some properties are easier in complex than real variables. This includes a look at their importance in solving polynomial equations. Addition and multiplications of complex numbers their representation helps us to understand the behavior of circuits which contain reactance when we apply A.C. signals. Electrical engineers use complex numbers to measure electrical current and to elucidate how electric circuits work. Mechanical engineers practice complex numbers to analyze the stresses of beams in buildings and bridges. With the help of eigenvalues and eigenvectors of the matrix the engineers configure to explain numerically the stresses of the beams. It gives us a new way to think about oscillations. Realizing the significance of complex numbers and it's applications in Euclid's plane geometry provides future teachers a strong analytical kit which can be used while working with students. On the other hand, Partial fraction decomposition is the process of taking a rational expression and disintegrating it into simpler rational expressions that we can add or subtract to get the original rational expression. Many integrals involving rational expressions can be done using partial fractions on the integrand.

PRE-REQUISITES

- Functions and their graphs
- Trigonometric function.
- Trigonometric identities
- Coordinate Geometry
- Familiarity with the algebraic techniques.
- Proper fraction.
- Improper fraction.
- Substitution.

UNIT OUTCOMES

List of outcomes of this unit are as follows:

U4-O1: Perform algebraic operations on complex numbers and plot complex numbers on an Argand diagram.

U4-O2: Employ Polar form [r, θ] of complex numbers and its algebra to solve given problems.

U 4-O3: Interpret the relationship of complex numbers as loci in the complex plane.

U 4-O4: Use de Movire's theorem for application-based problems.

U 4-O5: Express an algebraic fraction as the sum of its partial fractions.

Unit Outcome	Expected Mapping with Program Outcomes *(1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation)*						
	CO-1	CO-2	CO-3	CO-4	CO-5	CO-6	CO-7
U4-O1	-	-	-	1	1	1	3
U4-O2	-	1	-	1	2	1	3
U4-O3	-	1	-	3	1	1	2
U4-O4	-	1	-	2	1	1	3
U4-O5	-	-	-	2	2	1	-

4.1 DEFINITION AND ALGEBRA OF COMPLEX NUMBER

4.1.1 Basic Concept of Complex Number

Definition: A number of the form $x+iy$, where $x, y \in R$ and $i=\sqrt{-1}$ is called a complex number and 'i' is called iota.

A complex number is usually denoted by z and the set of complex number is denoted by C.

i.e., $C=\left\{x+iy : x \in R, y \in R, i=\sqrt{-1}\right\}$ *For example,* $5+3i, -1+i, 0+4i, 4+0i$ etc. are complex numbers.

(i) Euler was the first mathematician to introduce the symbol i (iota) for the square root of – 1 with property $i^2=-1$. He also called this symbol as the imaginary unit.

(ii) For any positive real number, a, we have

$\sqrt{-a}=\sqrt{-1\times a}=\sqrt{-1}\sqrt{a}=i\sqrt{a}$

(iii) The property $\sqrt{a}\sqrt{b}=\sqrt{ab}$ is valid only if at least one of a and b is non-negative. If a and b are both negative then $\sqrt{a}.\sqrt{b}=\sqrt{|a|.|b|}$.

Example 1: $\sqrt{-2}\sqrt{-3}=$

(a) $\sqrt{6}$ (b) $-\sqrt{6}$

(c) $i\sqrt{6}$ (d) None of these

Solution:

(b) $\sqrt{-2}\sqrt{-3}=i\sqrt{2}i\sqrt{3}=i^2\sqrt{6}=-\sqrt{6}$

Integral powers of *i* (iota):

Since $i=\sqrt{-1}$ hence we have $i^2=-1$, $i^3=-i$ and $i^4=1$. To find the value of $i^n\,(n>4)$, first divide n by 4. Let q be the quotient and r be the remainder.

i.e., $n=4q+r$ where $0\le r\le 3$

$i^n=i^{4q+r}=(i^4)^q.(i)^r=(1)^q.(i)^r=i^r$

In general, we have the following results $i^{4n}=1, i^{4n+1}=i$, $i^{4n+2}=-1$, $i^{4n+3}=-i$, where n is any integer.

Example 2: The value of $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1=$

(a) -1 (b) – 2

(c) -3 (d) – 4

Solution:

(b) $$\frac{i^{584}\left(i^8+i^6+i^4+i^2+1\right)}{i^{574}\left(i^8+i^6+i^4+i^2+1\right)}-1=\frac{i^{584}}{i^{574}}-1 \qquad =i^{10}-1=-1-1=-2$$

4.1.2 Real and Imaginary Parts of a Complex Number

If x and y are two real numbers, then a number of the form $z = x + iy$ is called a complex number. Here 'x' is called the real part of z and 'y' is known as the imaginary part of z. The real part of z is denoted by $Re(z)$ and the imaginary part by $\mathrm{Im}(z)$.

If $z = 3 - 4i$, then $\mathrm{Re}(z) = 3$ and $\mathrm{Im}(z) = -4$.

A complex number z is purely real if its imaginary part is zero *i.e.*, $\mathrm{Im}(z) = 0$ and purely imaginary if its real part is zero *i.e.*, $\mathrm{Re}(z) = 0$.

Example 3: $\mathrm{Re}(2i\text{-}3) = \ldots$

(a) -2 (b) 2

(c) -3 (d) 3

Solution:

(a) $\mathrm{Re}(2i\text{-}3) = \mathrm{Re}(-3+2i) = -3$

4.1.3 Algebraic Operations with Complex Numbers

Let two complex numbers *be* $z_1 = a + ib$ and $z_2 = c + id$

Addition ($z_1 + z_2$) : $(a+ib)+(c+id)=(a+c)+i(b+d)$

Subtraction $(z_1 - z_2)$: $(a+ib)-(c+id)=(a-c)+i(b-d)$

Multiplication $(z_1.z_2)$: $(a+ib)(c+id)=(ac-bd)+i(ad+bc)$

Division (z_1 / z_2) : $\dfrac{a+ib}{c+id}$

(where at least one of c and d is non-zero)

$$\frac{a+ib}{c+id} = \frac{(a+ib)}{(c+id)} . \frac{(c-id)}{(c-id)} \text{ (Rationalization)}$$

$$\frac{a+ib}{c+id} = \frac{(ac+bd)}{c^2+d^2} + \frac{i(bc-ad)}{c^2+d^2}.$$

4.1.4 Properties of Algebraic Operations on Complex Numbers

Let z_1, z_2 and z_3 are any three complex numbers then their algebraic operations satisfy following properties:

(i) Addition of complex numbers satisfies the commutative and associative properties

i.e., $z_1 + z_2 = z_2 + z_1$ and $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

(ii) Multiplication of complex numbers satisfies the commutative and associative properties.

i.e., $z_1 z_2 = z_2 z_1$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.

(iii) Multiplication of complex numbers is distributive over addition

i.e., $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ and $(z_2 + z_3) z_1 = z_2 z_1 + z_3 z_1$.

Example: 4: Perform the indicated operations and write the answer in the form $x+iy$ where $x, y \in R$.

(i) $(3-2i)+i(5+2i)$

(ii) $(\sqrt{2}-i)-(2+4i)+i(2+\sqrt{2}i)$

(iii) $(3-2i)(i+4)$

(iv) $(2-i)(2+i)(1+i)$

(v) $\dfrac{5-2i}{2+3i}$

(vi) $\dfrac{(2+i)(i-3)}{4i+3}$

(vii) $i^{12}+(3-2i)^2$

(viii) $(\sqrt{5}-7i)(\sqrt{5}+7i)^2$

(ix) $\dfrac{(\sqrt{2}-i\sqrt{3})+(\sqrt{2}+i\sqrt{3})}{1+i}$

Solution:

(i) $let, z=(3-2i)+i(5+2i)$

$$\therefore z=(3-2i)+(5i+2i^2)=3-2i+5i-2=1+3i$$

(ii) $let, z=(\sqrt{2}-i)-(2+4i)+i(2+\sqrt{2}i)$

$$z=(\sqrt{2}-i)-(2+4i)+i(2+\sqrt{2}i)=\sqrt{2}-i-2-4i+2i+\sqrt{2}\,i^2 \;=\sqrt{2}-3i-2-\sqrt{2}=-2-3i$$

(iii) $let, z=(3-2i)(i+4)$

$$\therefore z=(3-2i)(i+4)=3i+12-2i^2-8i=14-5i$$

(iv) $let, z=(2-i)(2+i)(1+i)$

$$\therefore z=(2-i)(2+i)(1+i)=((2)^2-(i)^2)(1+i)$$

$$=(4+1)(1+i)=5(1+i)=5+5i$$

(v) $let, z=\dfrac{5-2i}{2+3i}$

$$\therefore z=\frac{5-2i}{2+3i}=\frac{5-2i}{2+3i}\times\frac{2-3i}{2-3i}=\frac{10-15i-4i+6i^2}{(2)^2+(3)^2}=\frac{4-19i}{13}=\frac{4}{13}-\frac{19}{13}i$$

(vi) $let, z=\dfrac{(2+i)(i-3)}{4i+3}$

$$\therefore z=\frac{(2+i)(i-3)}{4i+3}=\frac{2i-6+i^2-3i}{3+4i}=\frac{-7-i}{3+4i}=\frac{-7-i}{3+4i}\times\frac{3-4i}{3-4i}$$

$$=\frac{-21+28i-3i+4i^2}{(3)^2+(4)^2}=\frac{-25+25i}{25}=-1+i$$

(vii) $let, z = i^{12} + (3-2i)^2$

$$z = i^{12} + (3-2i)^2 = 1 + 9 - 12i - 4 = 6 - 12i$$

(viii) $let, z = \left(\sqrt{5} - 7i\right)\left(\sqrt{5} + 7i\right)^2$

$$\therefore z = \left[\left(\sqrt{5}\right)^2 + (7)^2\right]\left(\sqrt{5} + 7i\right) = 54\left(\sqrt{5} + 7i\right) = 54\sqrt{5} + 378i$$

(ix) $let, z = \dfrac{\left(\sqrt{2} - i\sqrt{3}\right) + \left(\sqrt{2} + i\sqrt{3}\right)}{1+i}$

$$\therefore z = \frac{\left(\sqrt{2} - i\sqrt{3}\right) + \left(\sqrt{2} + i\sqrt{3}\right)}{1+i} = \frac{2\sqrt{2}}{1+i}$$

$$= \frac{2\sqrt{2}}{1+i} \times \frac{1-i}{1-i} = \frac{2\sqrt{2} - i2\sqrt{2}}{1+1} = \sqrt{2} - i\sqrt{2}$$

4.1.5 Equality of Two Complex Numbers

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if their real and imaginary parts are separately equal.

i.e., $z_1 = z_2 \Leftrightarrow x_1 + iy_1 = x_2 + iy_2 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$.

Complex numbers do not possess the property of order *i.e.,* $(a+ib)\langle(\text{or})\rangle(c+id)$ is not defined. For example, the statement $(9+6i) > (3+2i)$ makes no sense.

Example 5: The real values of x and y for which the equation is $(x+iy)\ (2-3i) = 4+i$ is satisfied, are

(a) $x = \dfrac{5}{13}, y = \dfrac{8}{13}$ (b) $x = \dfrac{8}{13}, y = \dfrac{5}{13}$

(c) $x = \dfrac{5}{13}, y = \dfrac{14}{13}$ (d) None of these

Solution:

(c) Equation $(x+iy)(2-3i) = 4+i$

$\Rightarrow (2x+3y) + i(-3x+2y) = 4+i$

Equating real and imaginary parts, we get

$2x + 3y = 4$ ......(i)

$-3x + 2y = 1$ ......(ii)

From (i) and (ii), we get $x = \frac{5}{13}, y = \frac{14}{13}$

Alter: $x + iy = \frac{4+i}{2-3i} = \frac{(4+i)(2+3i)}{13} = \frac{5}{13} + \frac{14}{13}i$

4.2 CONJUGATE OF A COMPLEX NUMBER

4.2.1 Conjugate Complex Number

If there exists a complex number $z = a + ib, a, b \in R$, then its conjugate is defined as $\overline{z} = a - ib$.

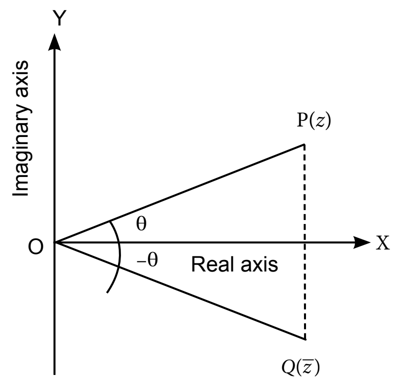

Fig. 4.1: Conjugate Complex Number

Hence, we have $\text{Re}(z) = \frac{z + \overline{z}}{2}$ and $Im(z) = \frac{z - \overline{z}}{2i}$.

Geometrically, the conjugate of z is the reflection or point image of z in the real axis.

4.2.2 Properties of Conjugate

If z, z_1 and z_2 are existing complex numbers, then we have the following results:

(i) $\overline{(\overline{z})} = z$

(ii) $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$

(iii) $\overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2$

(iv) $\overline{z_1 z_2} = \overline{z}_1\, \overline{z}_2$, In general, $\overline{z_1.z_2.z_3.....z_n} = \overline{z}_1.\overline{z}_2.\overline{z}_3.....\overline{z}_n$

(v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

(vi) $(\overline{z})^n = \left(\overline{z^n}\right)$

(vii) $z+\overline{z} = 2\,\text{Re}(z) = 2\,\text{Re}(\overline{z}) =$ purely real

(viii) $z-\overline{z} = 2iIm(z) =$ purely imaginary

(ix) $z\overline{z} = |z| =$ purely real

Example 6: The conjugate of the complex number $\dfrac{2+5i}{4-3i}$ is

(a) $\dfrac{7-26i}{25}$ (b) $\dfrac{-7-26i}{25}$

(c) $\dfrac{-7+26i}{25}$ (d) $\dfrac{7+26i}{25}$

Solution:

(b) $\dfrac{2+5i}{4-3i} = \dfrac{(2+5i)(4+3i)}{25} = \dfrac{-7+26i}{25}$.

Therefore, conjugate of the complex number is $\dfrac{-7-26i}{25}$.

Example 7: Find the complex conjugate of.

(i) $(-2+i)(3i-2)$ (ii) $\dfrac{1+7i}{(2-i)^2}$ (iii) $\dfrac{1+2i}{3-4i}+\dfrac{i-2}{3+4i}$

Solution:

(i) $let, z = (-2+i)(3i-2) = -6i+4+3i^2-2i = 1-8i$

$\therefore \overline{z} = 1+8i$

(ii) $let, z = \dfrac{1+7i}{(2-i)^2}$

$$\therefore z = \frac{1+7i}{(2-i)^2} = \frac{1+7i}{4-4i-1} = \frac{1+7i}{3-4i}$$

$$\therefore z = \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i+21i+28i^2}{(3)^2+(4)^2} = \frac{-25+25i}{25} = -1+i$$

$\therefore \overline{z} = -1-i$

(iii) $let, z = \dfrac{1+2i}{3-4i}+\dfrac{i-2}{3+4i}$

$$\therefore z = \frac{(1+2i)(3+4i)+(i-2)(3-4i)}{(3-4i)(3+4i)}$$

$$\therefore z = \frac{3+4i+6i+8i^2+3i-4i^2-6+8i}{9+16} = \frac{-7+21i}{25} = -\frac{7}{25}+\frac{21}{25}i$$

$$\therefore \overline{z} = -\frac{7}{25}-\frac{21}{25}i$$

4.3 MODULUS OF A COMPLEX NUMBER

4.3.1 Definition

Modulus of a complex number $z = a+ib$ is defined by a positive real number given by $|z| = \sqrt{a^2+b^2}$, where *a, b* real numbers. Geometrically $|z|$ represents the distance of point *P* from the origin, i.e.$|z| = OP$.

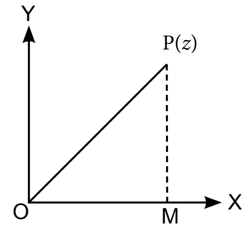

Fig. 4.2: Modules of Complex Number

If $|z|=1$ the corresponding complex number is known as **unimodular complex number.** Clearly *z* lies on a circle of unit radius having center (0, 0).

4.3.2 Properties of Modulus

(i) $|z| \geq 0 \Rightarrow |z| = 0 \text{ if} z = 0$ and $|z| > 0 \text{ if } z \neq 0$.

(ii) $-|z| \leq Re(z) \leq |z|$ and $-|z| \leq Im(z) \leq |z|$

(iii) $|z| = |\overline{z}| = |-z| = |-\overline{z}| = |zi|$

(iv) $z\overline{z} = |z|^2 = |\overline{z}|^2$

(v) $|z_1z_2| = |z_1||z_2|$.

In general, $|z_1z_2z_3......z_n| = |z_1||z_2||z_3|....|z_n|$

(vi) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, (z_2 \neq 0)$

(vii) $|z^n| = |z|^n, n \in N$

(viii) $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\overline{z}_1 \pm \overline{z}_2) = |z_1|^2 + |z_2|^2 \pm (z_1\overline{z}_2 + \overline{z}_1 z_2)$

or $|z_1|^2 + |z_2|^2 \pm 2\,\text{Re}(z_1\overline{z}_2)$

Example 8: Modulus of $\left(\frac{3+2i}{3-2i}\right)$ is

(a) 1 (b) 1/2

(c) 2 (d) $\sqrt{2}$

Solution:

(a) $\left(\dfrac{3+2i}{3-2i}\right)=\left(\dfrac{3+2i}{3-2i}\right)\left(\dfrac{3+2i}{3+2i}\right)=\dfrac{9-4+12i}{13}=\dfrac{5}{13}+i\left(\dfrac{12}{13}\right)$

Modulus $=\sqrt{\left(\dfrac{5}{13}\right)^2+\left(\dfrac{12}{13}\right)^2}=1$.

Example 9: Find the modulus (magnitude) of.

(i) $(2+5i)+3(i+1)-(1+5i)$ (ii) $\dfrac{\left(1+i\sqrt{2}\right)\left(1-i\sqrt{2}\right)}{4+3i}$

(iii) $4(1-i)+\left[(2+5i)(i-2)\right]$

Solution:

(i) $let, z=(2+5i)+3(i+1)-(1+5i)$

$\therefore z=(2+5i)+3(i+1)-(1+5i)=2+5i+3i+3-1-5i$

$\therefore z=4+3i$

$\therefore |z|=\sqrt{(4)^2+(3)^2}=\sqrt{25}=5$

(ii) $let, z=\dfrac{\left(1+i\sqrt{2}\right)\left(1-i\sqrt{2}\right)}{4+3i}$

$\therefore z=\dfrac{\left((1)^2+\left(\sqrt{2}\right)^2\right)}{4+3i}=\dfrac{3}{4+3i}$

$\therefore z=\dfrac{3}{4+3i}\times\dfrac{4-3i}{4-3i}=\dfrac{12-9i}{(4)^2+(3)^2}=\dfrac{12-9i}{25}=\dfrac{12}{25}-\dfrac{9}{25}i$

$|z|=\sqrt{\left(\dfrac{12}{25}\right)^2+\left(\dfrac{9}{25}\right)^2}=\sqrt{\dfrac{144+81}{625}}=\sqrt{\dfrac{225}{625}}=\dfrac{15}{25}=\dfrac{3}{5}$

OR

$\therefore z=\dfrac{3}{4+3i}=\dfrac{z_1}{z_2}$ where, $z_1=3=3+0i$ and $z_2=4+3i$

$\therefore |z_1|=\sqrt{(3)^2}=3$ and $|z_2|=\sqrt{(4)^2+(3)^2}=\sqrt{25}=5$

$\therefore \left|\dfrac{z_1}{z_2}\right|=\dfrac{|z_1|}{|z_2|}=\dfrac{3}{5}$

(iii) $let, z = 4(1-i) + [(2+5i)(i-2)]$

$$\therefore z = (4-4i) + (2i - 4 + 5i^2 - 10i) = 4 - 4i + 2i - 4 - 5 - 10i = -5 - 12i$$

$$\therefore z = -5 - 12i$$

$$\therefore |z| = \sqrt{(-5)^2 + (-12)^2} = \sqrt{25+144} = \sqrt{169} = 13$$

4.4 Argument (Amplitude) of a Complex Number

4.4.1 Definition

Let $z = a + ib$ be any complex number. If this complex number is represented geometrically by a point *P*, then the angle made by the line *OP* with real axis is known as argument or amplitude of *z* and is expressed as

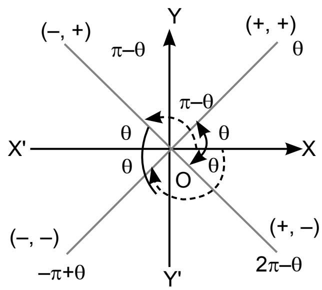

Fig. 4.3: Argument of Complex Number

$arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right), \theta = \angle POM$ also, argument of a complex number is not unique, since if θ be a value of the argument, so also is $2n\pi + \theta,$ where $n \in I$.

4.4.2 Principal value of *arg* (z)

The value θ of the argument, which satisfies the inequality $-\pi < \theta \leq \pi$ is called the principal value of argument, where $\alpha = \tan^{-1}\left|\frac{b}{a}\right|$ (acute angle) and principal values of argument *z* will be $\alpha, \pi - \alpha, -\pi + \alpha$ and $-\alpha$ as the point *z* lies in the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} quadrants respectively.

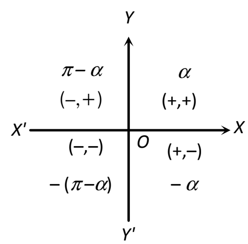

Fig. 4.4: Principal Value of Argument

Example 10: The argument of the complex number $-1 + i\sqrt{3}$ is

(a) –60° (b) 60°

(c) 120° (d) –120°

Solution:

(c) $arg\left(-1+i\sqrt{3}\right)=\tan^{1}\left(\frac{\sqrt{\ }}{\ }\right)=120$ because it lies in second quadrant.

Example 11: Find the argument of following.

(i) $1+i$ (ii) $-1-i\sqrt{3}$ (iii) $2+i\sqrt{3}$

Solution:

(i) $let, z=1+i$

$$\therefore z=1+i \quad here\, a=1 and\ b=1$$

$$\therefore \alpha=\tan^{-1}\left|\frac{b}{a}\right|=\left|\frac{1}{1}\right|=1\therefore \alpha=\frac{\pi}{4}$$

$here\ \ a>0\, and\, b>0 \therefore \theta lie\, in\, first\ \ quadrant$

$$\therefore amp\left(z\right)=\arg(z)=\theta=\alpha=\frac{\pi}{4}$$

(ii) $let, z=-1-i\sqrt{3}$

$$\therefore z=-1-i\sqrt{3}here a=-1 and b=-\sqrt{3}$$

$$\therefore \alpha=\tan^{-1}\left|\frac{b}{a}\right|=\left|\frac{-\sqrt{3}}{-1}\right|=\sqrt{3}\therefore \alpha=\frac{\pi}{3}$$

$here\, a<0\, and\, b<0 \therefore \theta lie\, in\, third\, quadrant$

$$\therefore amp\left(z\right)=\arg(z)=\theta=-\pi+\alpha=-\pi+\frac{\pi}{3}=-\frac{2\pi}{3}$$

(iii) $let, z=2+i\sqrt{3}$

$$\therefore z=2+i\sqrt{3} \qquad here\ \ a=2\ \ and\ \ b=\sqrt{3}$$

$$\therefore \alpha=\tan^{-1}\left|\frac{b}{a}\right|=\left|\frac{\sqrt{3}}{2}\right|=\frac{\sqrt{3}}{2}\therefore \alpha=\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$here\, a>0\, and\, b>0 \therefore \theta lie\, in\, first quadrant$

$$\therefore amp\left(z\right)=\arg(z)=\theta=\alpha=\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

Table 4.1: Value of arguments of complex numbers

Complex Number	Value of Arguments
+*ve* Re (*z*)	0
−*ve* Re (*z*)	π

Complex Number	Value of Arguments
$+ve$ Im (z)	$\pi/2$
$-ve$ Im (z)	$3\pi/2 or -\pi/2$
$-(z)$	$\lvert\theta \pm \pi\rvert, if\, \theta is -ve and +ve$ respectively
(iz)	$\left\{\frac{\pi}{2} + arg(z)\right\}$
$-(iz)$	$\left\{arg(z) \quad -\right\}$
$\left(z^n\right)$	$n.arg$ (z)
$\left(z_1.z_2\right)$	arg (z_1) + arg (z_2)
$\left(\frac{z_1}{z_2}\right)$	arg (z_1) – arg (z_2)

4.5 VARIOUS REPRESENTATIONS OF A COMPLEX NUMBER

A complex number can be represented in the following form:

4.5.1 Geometrical Representation (Cartesian Representation)

The complex number $z = a + ib = (a, b)$ is represented by a point P whose coordinates are referred to rectangular axes XOX' and YOY' which are called real and imaginary axis respectively. This plane is called argand plane or argand diagram or complex plane or Gaussian plane.

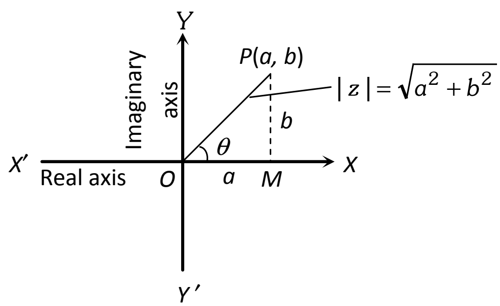

Fig. 4.5: Cartesian Representation of Complex Number

Distance of any complex number from the origin is called the modulus of complex number and is denoted by $|z|$, *i.e.*, $|z| = \sqrt{a^2 + b^2}$.

Angle of any complex number with positive direction of x–axis is called amplitude or argument of z.

i.e., $amp(z) = arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$

4.5.2 Trigonometrical (Polar) Representation

InΔOPM, let $OP = r$, then $a = r\cos\theta$ and $b = r\sin\theta$. Hence z can be expressed as $z = r(\cos\theta + i\sin\theta)$

where r = $|z|$ and θ = principal value of argument of z.

For general values of the argument

$$z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$$

Sometimes $(\cos\theta + i\sin\theta)$ is written in short as $cis\theta$.

Complex Numbers

4.5.3 Conversion of One form to Another

Example 12: Polar form of $\frac{1-i}{1+i}$ is equal to

(a) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ (b) $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

(c) $\sin\frac{\pi}{2} + i\cos\frac{\pi}{2}$ (d) None of these

Solution:

(b) $\frac{1-\boldsymbol{i}}{1+\boldsymbol{i}} = \frac{(1-\boldsymbol{i})(1-\boldsymbol{i})}{(1+\boldsymbol{i})(1-\boldsymbol{i})} = \frac{1+(\boldsymbol{i})^2 - 2\boldsymbol{i}}{1+1} = -\boldsymbol{i}$

which can be written as $\cos\frac{\pi}{2} - i\ \sin\frac{\pi}{2}$

Example: 13: Convert Cartesian to Polar form of the following complex numbers.

(i) i (ii) $-1+i$ (ii) $2\sqrt{3} - 2i$

Solution:

(i) $let, z = i$

$$\therefore z = i = 0 + i \ here \ \ a = 0 \ and \ b = 1$$

$$\therefore r = \sqrt{a^2 + b^2} = \sqrt{0+1} = 1$$

$$here \ a = 0 \ and \ b > 0$$

$$\therefore amp(z) = \arg(z) = \theta = \alpha = \frac{\pi}{2}$$

$$\therefore z = rcis\theta = r(\cos\theta + i\sin\theta) = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

(ii) $let, z = -1 + i$

$\therefore z = -1 + i\, here\ a = -1 and\ b = 1$

$\therefore r = \sqrt{a^2 + b^2} = \sqrt{1+1} = \sqrt{2}$

$\therefore \alpha = \tan^{-1}\left|\frac{b}{a}\right| = \left|\frac{1}{-1}\right| = 1 \therefore \alpha = \frac{\pi}{4}$

here $a < 0$ and $b > 0$, θ lie in second quadrant

$\therefore amp(z) = \arg(z) = \theta = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\therefore z = rcis\theta = r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(iii) $let, z = 2\sqrt{3} - 2i$

$\therefore z = 2\sqrt{3} - 2i\ here\ a = 2\sqrt{3} and\ b = -2$

$\therefore r = \sqrt{a^2 + b^2} = \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2} = \sqrt{12+4} = 4$

$\therefore \alpha = \tan^{-1}\left|\frac{b}{a}\right| = \left|\frac{-2}{2\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \therefore \alpha = \frac{\pi}{6}$

$here\, a > 0\, and\, b < 0, \theta\, lie\, in\ fourth\ quadrant$

$\therefore amp(z) = \arg(z) = \theta = -\alpha = -\frac{\pi}{6}$

$\therefore z = rcis\theta = r(\cos\theta + i\sin\theta) = 4\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right) = 4\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

4.6 DE' MOIVRE'S THEOREM

(1) If n is any rational number, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

(2) If $z = (cos\theta_1 + isin\theta_1)(cos\theta_2 + isin\theta_2)(cos\theta_3 + isin\theta_3)\ ......\ (cos\theta_n + isin\theta_n)$

then $z = \cos(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n)\ + i\sin(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n)$

where $\theta_1, \theta_2, \theta_3 ..... \theta_n \in R$.

(3) If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer,

then $z^{1/n} = r^{1/n}\left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right]$, where $k = 0, 1, 2, 3, \dots (n-1)$.

This theorem is not valid when n is not a rational number or the complex number is not in the form of $\cos\theta + i\sin\theta$.

Example 14: If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then

(a) $\text{Re}(z) = 0$ (b) $Im(z) = 0$

(c) $\text{Re}(z) > 0, Im(z) > 0$ (d) $\text{Re}(z) > 0, Im(z) < 0$

Solution:

(a) Using De Moivre's theorem

$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$ and putting $n = 0$, 1, 2 then we get required roots.

Example 15: Using De Moivre's theorem simplify the following.

(i) $\left[\frac{\cos 3\theta + i\sin 3\theta}{\cos\theta - i\sin\theta}\right]^2$

(ii) $\frac{(\cos 3\theta + i\sin 3\theta)^2 (\cos 5\theta + i\sin 5\theta)^{-3}}{(\cos 4\theta + i\sin 4\theta)^{-5} (\cos 2\theta + i\sin 2\theta)^4}$

Solution:

(i) *let* $z = \left[\frac{\cos 3\theta + i\sin 3\theta}{\cos\theta - i\sin\theta}\right]^2$

$$\therefore z = \frac{(\cos 3\theta + i\sin 3\theta)^2}{(\cos\theta - i\sin\theta)^2} = \frac{\left[(\cos\theta + i\sin\theta)^3\right]^2}{\left[(\cos\theta + i\sin\theta)^{-1}\right]^2} = \frac{(\cos\theta + i\sin\theta)^6}{(cos\theta + i\sin\theta)^{-2}}$$

$$\therefore z = (\cos\theta + i\sin\theta)^{6-(-2)} = (\cos\theta + i\sin\theta)^8 = \cos 8\theta + i\sin 8\theta$$

(ii) *let* $z = \frac{(\cos 3\theta + i\sin 3\theta)^2 (\cos 5\theta + i\sin 5\theta)^{-3}}{(\cos 4\theta + i\sin 4\theta)^{-5} (\cos 2\theta + i\sin 2\theta)^4}$

$$\therefore z = \frac{\left[(\cos\theta + i\sin\theta)^3\right]^2 \left[(\cos\theta + i\sin\theta)^5\right]^{-3}}{\left[(\cos\theta + i\sin\theta)^4\right]^{-5} \left[(\cos\theta + i\sin\theta)^2\right]^4} = \frac{(\cos\theta + i\sin\theta)^6 (\cos\theta + i\sin\theta)^{-15}}{(\cos\theta + i\sin\theta)^{-20} (\cos\theta + i\sin\theta)^8}$$

$$\therefore z == (\cos\theta + i\sin\theta)^{6+(-15)-(-20)-8} = (\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

Example 16: Prove that $\left(\sin\theta+i\cos\theta\right)^{4n}=cos4n\theta-i\sin4n\theta, n\in N$

Solution:

$$L.H.S.=\left(\sin\theta+i\cos\theta\right)^{4n}=\left[\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right]^{4n}$$

$$=\cos4n\left(\frac{\pi}{2}-\theta\right)+i\sin4n\left(\frac{\pi}{2}-\theta\right)=\cos\left(2n\pi-4n\theta\right)+i\sin\left(2n\pi-4n\theta\right)$$

$$=\cos\left(4n\theta\right)-i\sin\left(4n\theta\right)=R.H.S.$$

Example 17: $Prove\,that\left(\sqrt{3}+i\right)^{n}+\left(\sqrt{3}-i\right)^{n}=2^{n+1}\cos\left(\frac{n\pi}{6}\right), n>0$

Solution:

$$Let\ z_1=\left(\sqrt{3}+i\right)and z_2=\left(\sqrt{3}-i\right)$$

By converting $z_1 and z_2$ in to polar form we get

$$\therefore z_1=\left(\sqrt{3}+i\right)=2\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]and z_2=\left(\sqrt{3}-i\right)=2\left[\cos\left(\frac{\pi}{6}\right)-i\sin\left(\frac{\pi}{6}\right)\right]$$

$$Now, L.H.S.=\left(\sqrt{3}+i\right)^{n}+\left(\sqrt{3}-i\right)^{n}$$

$$=\left\{2\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]\right\}^{n}+\left\{2\left[\cos\left(\frac{\pi}{6}\right)-i\sin\left(\frac{\pi}{6}\right)\right]\right\}^{n}$$

$$=2^{n}\left[\cos\left(\frac{n\pi}{6}\right)+i\sin\left(\frac{n\pi}{6}\right)+\cos\left(\frac{n\pi}{6}\right)-i\sin\left(\frac{n\pi}{6}\right)\right]$$

$$=2^{n+1}\cos\left(\frac{n\pi}{6}\right)=R.H.S.$$

More on Complex nos

4.7 PARTIAL FRACTIONS

Recall that a rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where P (x) and Q(x) are polynomials in x and Q(x) ≠ 0. If the degree of P(x) is less than the degree of Q(x), then the rational function is called proper, otherwise, it is called improper. The improper rational functions can be reduced to the proper rational functions by long division process. Thus, if $\frac{P(x)}{Q(x)}$. is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$ where T(x) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is a proper rational function. The following Table indicates the types of simpler partial fractions that are to be associated with various kind of rational functions.

Table 4.2: Forms of rational functions and respective partial fraction

S. No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+C)}$	$\frac{A}{x-a} + \frac{Bx+C}{(x^2+bx+C)}$
Where x^2+bx+C can not be factorized further.		

Example 18: Resolve into partial Fraction $\frac{1}{(x+1)(x+2)}$

Solution:

The integrand is a proper rational function. Therefore, by using the form of partial fraction [Table], we write

$$\frac{1}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

where, real numbers A and B are to be determined suitably. This gives

1 = A (x + 2) + B (x + 1).

Equating the coefficients of x and the constant term, we get

A + B = 0 and

2A + B = 1

Solving these equations, we get A =1 and B = – 1.

Thus, fraction is $\frac{1}{(x+1)(x+2)} = \frac{1}{(x+1)} + \frac{-1}{(x+2)}$

Example 19: Resolve into partial Fraction $\frac{x^2+1}{x^2-5x+6}$

Solution:

Here the fraction is not proper rational function, so we divide x^2+1 by $x^2 - 5x+6$ and find that

$$\frac{x\ \ +1}{x^2-5x+6}=1+\frac{5x-5}{x^2-5x+6}=1+\frac{5x-5}{(x-2)(x-3)}$$

Let $\frac{5x-5}{(x-2)(x-3)}=\frac{A}{x-2}+\frac{B}{x-3}$

So that $5x-5=A(x-3)+B(x-2)$

Equating the coefficients of x and constant terms on both sides, we get
A + B = 5 and
3A + 2B = 5.
Solving these equations, we get A = – 5 and B = 10
So, we have

$$\frac{x^2+1}{x^2-5x+6}=1-\frac{5}{x-2}+\frac{10}{x-3}$$

Example 20: Resolve into partial Fraction $\frac{3x-2}{(x+1)^2(x+3)}$

Solution:

The integrand is of the type as given in Table. We write

$$\frac{3x-2}{(x+1)^2(x+3)}=\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x+3)}$$

So that

$$3x-2=A(x+1)(x+3)+B(x+3)+C(x+1)^2$$

$$3x-2=A\left(x^2+4x+3\right)+B(x+3)+C\left(x^2+2x+1\right)$$

Comparing coefficient of x^2, x and constant term on both sides, we get

$A+C=0$,

$4A+B+2C=3$ and

$3A+3B+C=-2$.

Solving these equations, we get $A=\frac{11}{4}, B=-\frac{5}{2} and\, C=-\frac{11}{4}$

Thus, we have,

$$\frac{3x-2}{(x+1)^2(x+3)}=\frac{11}{4(x+1)}-\frac{5}{2(x+1)^2}-\frac{11}{4(x+3)}$$

Example 21: Resolve into partial Fraction $\frac{x^2}{(x^2+1)(x^2+4)}$

Solution:

Consider $\frac{x^2}{(x^2+1)(x^2+4)}$ and put $x^2=y$

Then we have

$$\frac{x^2}{(x^2+1)(x^2+4)}=\frac{y}{(y+1)(y+4)}$$

Now $\frac{y}{(y+1)(y+4)}=\frac{A}{y+1}+\frac{B}{y+4}$

So that $y=A(y+4)+B(y+1)$

Comparing coefficients of y and constant terms on both sides, we get

A + B = 1 and

4A + B = 0,

which give $A=-\frac{1}{3},B=\frac{4}{3}$. So, we have

$$\frac{y}{(y+1)(y+4)}=-\frac{1}{3(y+1)}+\frac{4}{3(y+4)}$$

Or

$$\frac{x^2}{(x^2+1)(x^2+4)}=-\frac{1}{3(x^2+1)}+\frac{4}{3(x^2+4)}$$

Example 22: Resolve into partial Fraction $\frac{x^2+x+1}{(x+2)(x^2+1)}$

Solution:

Decompose the rational function into partial fraction [Ref Table]. Write

$$\frac{x^2+x+1}{(x+2)(x^2+1)}=\frac{A}{x+2}+\frac{Bx+c}{x^2+1}$$

Therefore, $x^2+x+1=A(x^2+1)+(Bx+C)(x+2)$

Equating the coefficients of $x^2, x\, and\, constant\, term$ of both sides, we get

A + B =1,
2B + C = 1 and
A + 2C = 1.

Solving these equations, we get $A=\frac{3}{5}, B=\frac{2}{5}\, and\, C=\frac{1}{5}$

So, we have

$$\frac{x^2+x+1}{(x+2)(x^2+1)}=\frac{3}{5(x+2)}+\frac{\frac{2}{5}x+\frac{1}{5}}{x^2+1}=\frac{3}{5(x+2)}+\frac{2x+1}{5(x^2+1)}$$

APPLICATIONS (REAL LIFE / INDUSTRIAL)

Control Theory

Example 1: When the system in the control theory is transformed from the time domain to the frequency domain then the role of the Laplace transform starts, with this transform stability of the system is analyzed on the basis of theory of poles and zeros in the complex plane.

Residue Theorem

Example 2: Evaluation of path integrals of meromorphic functions is based on the residue theorem in complex analysis, this Theorem also helps to compute real integrals.

Electromagnetism

Example 3: In Electromagnetism the real part can be taken as electrical and magnetic part can be taken as imaginary and as a whole this can be taken as one complex number.

Civil and Mechanical Engineering

Example 4: The idea of complex geometry and Argand plane is very much useful in making buildings and cars. It is also very suitable in cutting of tools.

Fluid Dynamics

Example 5: Analytic functions are used to describe potential flow in two dimensions they are also used in calculating forces and moments and prediction of weather patterns.

Geometry

Example 6: Concept of the Complex numbers is used in Fractals (e.g. the Mandelbrot set and Julia sets).

Electrical Engineering

Example 7: A 2-dimensional quantity can be represented mathematically by a complex number. In the complex number representation, the components are referred to as real and imaginary for example the "AC" voltage in a home requires two parameters.

Signal Analysis

Example 8: Complex numbers are used in signal analysis. For a sine wave of a given frequency, the absolute value |z| of the corresponding z is the amplitude and the argument arg (z) the phase.

Rational Functions in Calculus

Example 9: By doing partial fractions on the integrand, many integrals involving rational expressions can be done.

Differential Equations

Example 10: Partial fractions can be utilized to find the Inverse Laplace Transform in the theory of differential equations.

UNIT SUMMARY

In this unit the first section is devoted to define real and imaginary parts of a Complex number, along with representation in polar and Cartesian forms and its conversion from one form to other. Subsequent sections deal with Conjugate of a complex number, modulus and amplitude of a complex numbers and basic arithmetic with complex numbers, more precisely definition of subtraction and division. For example, multiplication can be described geometrically. Algebraic and geometric properties are also discussed. Last three sections are based on definition of polynomial fraction proper & improper fractions and definition of partial fractions and applications. New examples designed to reinforce the main concepts and applications of complex numbers. Some open questions are also advised these Verbal questions will help for assessing conceptual understanding of key terms and concepts. On the other hand, Algebraic problems will help students to apply algebraic manipulations, problems related with product of complex numbers assess students' ability to interpret the same from graphical approach. Numeric problems require the student perform calculations or computations. Real-World Applications present realistic problem scenarios.

EXERCISES

Multiple Choice Questions

1. If n is a positive integer, then which of the following relations is false

(a) $i^{4n} = 1$ (b) $i^{4n-1} = i$

(c) $i^{4n+1} = i$ (d) $i^{-4n} = 1$

2. If n is a positive integer, then $\left(\frac{1+i}{1-i}\right)^{4n+1} =$

(a) 1 (b) – 1

(c) i (d) $-i$

3. If $\left(\frac{1+i}{1-i}\right)^m = 1$, then the least integral value of m is

(a) 2 (b) 4
(c) 8 (d) None of these

4. If $(1-i)^n = 2^n$, then $n =$

(a) 1 (b) 0
(c) -1 (d) None of these

5. The value of $(1+i)^5 \times (1-i)^5$ is

(a) – 8 (b) $8i$
(c) 8 (d) 32

6. $\left(\frac{1+i}{1-i}\right)^2 + \left(\frac{1-i}{1+i}\right)^2$ is equal to

(a) $2i$ (b) $-2i$
(c) -2 (d) 2

7. $1 + i^2 + i^4 + i^6 + ... + i^{2n}$ is

(a) Positive (b) Negative
(c) Zero (d) Cannot be determined

8. $i^2 + i^4 + i^6 + ...$ up to $(2n+1)$ terms =

(a) i (b) $-i$
(c) 1 (d) -1

9. If $i = \sqrt{-1}$, then $1 + i^2 + i^3 - i^6 + i^8$ is equal to

(a) $2-i$ (b) 1
(c) 3 (d) -1

10. If $i^2 = -1$, then the value of $\sum_{n=1}^{200} i^n$ is

(a) 50 (b) – 50
(c) 0 (d) 100

11. The value of the sum $\sum_{n=1}^{13} \left(i^n + i^{n+1}\right)$, where $i = \sqrt{-1}$, equals

(a) i (b) $i-1$
(c) $-i$ (d) 0

12. The least positive integer n which will reduce $\left(\frac{i-1}{i+1}\right)^n$ to a real number, is

(a) 2 (b) 3
(c) 4 (d) 5

13. The value of $i^n + i^{n+1} + i^{n+2} + i^{n+3}, (n \in N)$ is

(a) 0 (b) 1
(c) 2 (d) None of these

14. The value of $(1+i)^8 + (1-i)^8$ is

(a) 16 (b) – 16
(c) 32 (d) – 32

15. $(1+i)^{10}$, where $i^2 = -1$, is equal to

(a) 32 i (b) 64 + i
(c) 24 i – 32 (d) None of these

Problems Based on Conjugate, Modulus and Argument of Complex Numbers

16. If $(a+ib)(c+id)(e+if)(g+ih) = A+iB$, then $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) =$

(a) A^2+B^2 (b) A^2-B^2
(c) A^2 (d) B^2

17. For the complex number z, one from $z+\overline{z}$ and $z\overline{z}$ is
(a) A real number (b) An imaginary number
(c) Both are real numbers (d) Both are imaginary numbers

18. The values of x and y for which the numbers $3+ix^2y$ and x^2+y+4i are conjugate complex can be

(a) $(-2,-1)$ or $(2,-1)$ (b) $(-1,2)$ or $(-2,1)$
(c) $(1,2)$ or $(-1,-2)$ (d) None of these

19. If $z = 3+5i$, then $z^3 + \overline{z} + 198 =$

(a) $-3-5i$ (b) $-3+5i$
(c) $3+5i$ (d) $3-5i$

20. The conjugate of complex number $\frac{2-3i}{4-i}$, is

(a) $\frac{3i}{4}$ (b) $\frac{11+10i}{17}$

(c) $\frac{11-10i}{17}$ (d) $\frac{2+3i}{4i}$

21. Conjugate of $1 + i$ is

(a) i (b) 1

(c) $1 - i$ (d) $1 + i$

22. $\left|(1+i)\frac{(2+i)}{(3+i)}\right| =$

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$

(c) 1 (d) -1

23. $arg\left(5-\sqrt{3}i\right)=$

(a) $\tan^{-1}\frac{5}{\sqrt{3}}$ (b) $\tan^{-1}\left(-\frac{5}{\sqrt{3}}\right)$

(c) $\tan^{-1}\frac{\sqrt{3}}{5}$ (d) $\tan^{-1}\left(-\frac{\sqrt{3}}{5}\right)$

24. Argument and modulus of $\frac{1+i}{1-i}$ are respectively

(a) $\frac{-\pi}{2}$ and 1 (b) $\frac{\pi}{2}$ and $\sqrt{2}$

(c) 0 and $\sqrt{2}$ (d) $\frac{\pi}{2}$ and 1

25. $arg\left(\frac{3+i}{2-i}+\frac{3-i}{2+i}\right)$ is equal to

(a) $\frac{\pi}{2}$ (b) $-\frac{\pi}{2}$

(c) 0 (d) $\frac{\pi}{4}$

26. If $x+iy=\sqrt{\frac{a+ib}{c+id}}$, then

(a) $\frac{a^2+b^2}{c^2+d^2}$ (b) $\frac{a+b}{c+d}$

(c) $\dfrac{c^2+d^2}{a^2+b^2}$ (d) $\left(\dfrac{a^2+b^2}{c^2+d^2}\right)^2$

27. The number of non-zero integral solutions of the equation $|1-i|^x = 2^x$ is

(a) Infinite (b) 1

(c) 2 (d) None of these

Problems based on De Moivre's Theorem

28. $\sqrt{i} =$

(a) $\dfrac{1 \pm i}{\sqrt{2}}$ (b) $\pm\dfrac{1-i}{\sqrt{2}}$

(c) $\pm\dfrac{1+i}{\sqrt{2}}$ (d) None of these

29. $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4$ equals

(a) $\sin 8\theta - i\cos 8\theta$ (b) $\cos 8\theta - i\sin 8\theta$

(c) $\sin 8\theta \quad \cos 8\theta$ (d) $\cos 8\theta + i\sin 8\theta$

30. $(-\sqrt{3}+i)^{53}$ where $i^2 = -1$ is equal to

(a) $2^{53}\left(\sqrt{3}+2i\right)$ (b) $2^{52}\left(\sqrt{3}-i\right)$

(c) $2^{53}\left(\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i\right)$ (d) $2^{53}\left(\sqrt{3}-i\right)$

Answers of Multiple-Choice Questions									
1.	b	**2.**	c	**3.**	b	**4.**	b	**5.**	d
6.	c	**7.**	d	**8.**	d	**9.**	a	**10.**	c
11.	b	**12.**	a	**13.**	a	**14.**	c	**15.**	a
16.	a	**17.**	c	**18.**	a	**19.**	c	**20.**	b
21.	c	**22.**	c	**23.**	d	**24.**	d	**25.**	c
26.	a	**27.**	d	**28.**	c	**29.**	d	**30.**	c

Short and Long Answer Type Questions

1. Find the Real and Imaginary part of

(i) $\sqrt{-16}+\sqrt{-3}$ (ii) $\dfrac{3}{2}i-\dfrac{1}{\sqrt{2}}$ (ii) $2+i-\sqrt{3}i$

2. Perform the indicated operations and write the answer in the form $x+iy$ where $x, y \in R$.

(i) $(3+2i)^3$ (ii) $(\sqrt{2}+3i)(\sqrt{2}-3i)^2+(3+2i)$

(iii) $\dfrac{(2+3i)(1-i)}{(1+2i)(2+i)}$ (iv) $\left(\dfrac{i}{3}+2\right)^2$

(v) $(2+i\sqrt{3})\left(\dfrac{1}{\sqrt{7}}+\dfrac{1}{\sqrt{7}}i\right)(2-i\sqrt{3})$ (vi) $\dfrac{(1-i)}{(1+i)^2}$

3. Find the modulus (magnitude) of the following.

(i) $-2+i\sqrt{5}$ (ii) $\dfrac{1}{2}+3i$ (ii) $(1+i)(17+7i)$

4. Find the complex conjugate of the following.

(i) $\dfrac{(2-3i)(6-i)}{(3+4i)}$ (ii) $\dfrac{-3+2i}{1-i}$ (iii) $\dfrac{2-\sqrt{-25}}{1-\sqrt{-16}}$

5. Find the multiplicative inverse of.

(i) $(1-3i)^2$ (ii) $\dfrac{4+3i}{5-3i}$ (iii) $\sqrt{5}+3i$

6. Find the argument (amplitude) of the following.

(i) $2+i$ (ii) $3i$ (ii) $\sqrt{3}+i$

7. Convert Cartesian to Polar form of complex number.

(i) i (ii) $-1+i$ (ii) $2\sqrt{3}-2i$

8. For the complex numbers $z_1 \, and \, z_2$ *prove that*,

(i) $\text{Re}(z_1+z_2)=\text{Re}(z_1)+\text{Re}(z_2)$

(ii) $Im(z_1 \cdot z_2)=\text{Re}(z_1)Im(z_2)+Im(z_1)\text{Re}(z_2)$

(iii) $(z_1+z_2)^2=(z_1)^2+2z_1z_2+(z_2)^2$

9. For $z_1=2+3i$ and $z_2=5+12i$, verify $\left|\dfrac{z_1}{z_2}\right|=\dfrac{|z_1|}{|z_2|}$

10. *If* $1+i4\sqrt{3}=(b+ai)^2$ *then prove that* $b^2-a^2=1$ *and* $ab=2\sqrt{3}.$

11. Show that

If z is any complex number with $|z|=1$, *then* $\dfrac{z-1}{z+1}$ is either zero or purely imaginary.

12. For any complex number z, prove that arg $\overline{z} = 2\pi - \arg z, z \neq 0$

13. Using Moivre's theorem simplify the following.

(i) $$\frac{(\cos\theta + i\sin\theta)^6 (\cos 5\theta + i\sin 5\theta)}{(\cos 4\theta + i\sin 4\theta)^2}$$

(ii) $$\frac{(\cos 4\theta + i\sin 4\theta)^2 (\cos 3\theta - i\sin 3\theta)^3}{(\cos 2\theta - i\sin 2\theta)^3 (\cos 5\theta + i\sin 5\theta)}$$

14. Prove that $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1} \cos\left(\frac{n\pi}{4}\right), n > 0$

15. Prove that $(1+\cos\theta + i\sin\theta)^n = 2^n \cos^n\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{n\theta}{2}\right) + i\sin\left(\frac{n\theta}{2}\right)\right], n > 0$

16. Prove that $\dfrac{1-\cos\theta + i\sin\theta}{1+\cos\theta + i\sin\theta} = ie^{-i\theta}\tan\left(\frac{\theta}{2}\right)$

17. Resolve into partial Fraction $\dfrac{x-1}{(x+1)(x-2)}$

18. Resolve into partial Fraction $\dfrac{2x-1}{(x-1)(x+2)(x-3)}$

19. Resolve into partial Fraction $\dfrac{3x+2}{(x-1)(x-2)(x-3)}$

20. Resolve into partial Fraction $\dfrac{x^3 - 6x^2 + 10x - 2}{x^2 - 5x + 6}$

21. Resolve into partial Fraction $\dfrac{3x+1}{(x-2)^2 (x+2)}$

22. Resolve into partial Fraction $\dfrac{x^2+1}{(x-1)^2 (x+3)}$

23. Resolve into partial Fraction $\dfrac{2x-3}{(x-1)^2 (x+1)(x+2)}$

24. Resolve into partial Fraction $\dfrac{2x-1}{(x+1)(x^2+2)}$

25. Resolve into partial Fraction $\frac{8}{(x+2)(x^2+4)}$

26. Resolve into partial Fraction $\frac{x}{(x-1)(x^2+4)}$

27. Resolve into partial Fraction $\frac{x^2+3x+1}{(x^2+1)(x^2+2)}$

Answers of Short and Long Answer Type Questions

1. (i) $\text{Re}(z)=0$ and $Im(z)=4+\sqrt{3}$

(ii) $\text{Re}(z)=\frac{-1}{\sqrt{2}}$ and $Im(z)=\frac{3}{2}$

(iii) $\text{Re}(z)=2$ and $Im(z)=1-\sqrt{3}$

2. (i) $-9+46i$ (ii) $(11\sqrt{2}+3)-31i$ (iii) $\frac{1}{5}-i$ (iv) $\frac{35}{9}+\frac{4}{3}i$

(v) $\sqrt{7}+i\sqrt{7}$ (vi) $-\frac{1}{2}-\frac{1}{2}i$

3. (i) 3 (ii) $\frac{\sqrt{37}}{2}$ (iii) 26

4. (i) $-\frac{7}{5}+\frac{24}{5}i$ (ii) $-\frac{5}{2}-\frac{1}{2}i$ (iii) $\frac{22}{17}-\frac{3}{17}i$

5. (i) $\frac{2}{25}+\frac{3}{50}i$ (ii) $\frac{11}{25}-\frac{27}{25}i$ (iii) $\frac{\sqrt{5}}{14}-\frac{3}{14}i$

6. (i) $\tan^{-1}\left(\frac{1}{2}\right)$ (ii) $\frac{\pi}{2}$ (iii) $\frac{\pi}{6}$

7. (i) $\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$ (ii) $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ (iii) $4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

10. True

13. (i) $\cos 3\theta+i\sin 3\theta$ (ii) 1

17. $\frac{2}{3(x+1)}+\frac{1}{3(x-2)}$

Answers of Short and Long Answer Type Questions
18. $-\frac{1}{6(x-1)}-\frac{1}{3(x+2)}+\frac{1}{2(x-3)}$
19. $\frac{5}{2(x-1)}-\frac{8}{x-2}+\frac{11}{2(x-3)}$
20. $(x-1)-\frac{2}{x-2}+\frac{1}{x-3}$
21. $\frac{5}{16(x-2)}+\frac{7}{4(x-2)^2}-\frac{5}{16(x+2)}$
22. $\frac{3}{8(x-1)}+\frac{1}{2(x-1)^2}+\frac{5}{8(x+3)}$
23. $\frac{17}{36(x-1)}-\frac{1}{6(x-1)^2}-\frac{5}{4(x+1)}+\frac{7}{9(x+2)}$
24. $-\frac{1}{x+1}+\frac{x+1}{x^2+2}$
25. $\frac{1}{x+2}+\frac{-x+2}{x^2+4}$
26. $\frac{1}{5(x-1)}+\frac{-x+4}{5(x^2+4)}$
27. $\frac{3x}{(x^2+1)}+\frac{-3x+1}{(x^2+2)}$

KNOW MORE

- Why Study Complex numbers?
- How Complex numbers and Partial fractions are used in our daily lives.
- Realize the need to expand the set of real numbers.
- Adopt the notions complex plain and Riemann sphere.
- Why mathematical thinking is valuable in daily life
- Shift to Online Teaching.
- The simplest way to learn Complex numbers and its algebra.

- Why was Theory of Complex variable invented?
- Learning of Complex analysis intuitively.
- Making of calculations less complicated.
- Online Education Tools for Teachers.
- Teaching Critical Thinking
- STEM Education.

Mini Project

i. Produce a mature oral presentation of a non-trivial mathematical topic based on concept of complex numbers.

ii. Prepare a case study on the comment "people thought of complex numbers as unreal, imagined.

Inquisitiveness and Curiosity Topics

i. How a real-life quantity can be described naturally by complex numbers rather than real number?

ii. What happens if we try to find square roots of negative numbers?

iii. If the system of real numbers be extended to a new system of numbers whether it is necessarily complex numbers?

iv. How and why a specific job uses the square roots of negative numbers?

v. What kind of relationship can be made between geometry and Imaginary Numbers?

vi. Whether a component in an electronic circuit can be measured by a complex number?

Apart from the above questions, concept of complex numbers is used to Geometry, Algebraic number theory, Analytic number theory, Improper integrals, Dynamic equations, applied mathematics and to physics.

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5 Permutation and Combination, Binomial Theorem

UNIT SPECIFICS

This unit elaborately discusses the following topics:

- Permutations and Combinations;
- Value of ${}^nP_r$ and ${}^nC_r$;
- Binomial theorem (without proof) for positive integral index;
- Binomial theorem for any index (expansion without proof);
- First and second binomial approximation with applications to engineering problems.

The applications-based problems are discussed for generating further curiosity and creativity as well as improving problem solving capacity.

Besides giving a large number of multiple-choice questions as well as questions of short and long answer types marked in two categories following lower and higher order of Bloom's taxonomy, assignments through a number of numerical problems, a list of references and suggested readings are given in the unit so that one can go through them for practice.

Based on the content, there is "Know More" section added. This section has been thoughtfully planned so that the supplementary information provided in this part becomes beneficial for the users of the book. This section mainly highlights further teaching and learning related to some interesting facts about why study Permutations and Combinations?, how Binomial theorem and its coefficients are used in our daily lives, Use of mathematical reasoning by justifying and generalizing patterns and relationships, Development of mathematics in historical context with contemporary non-mathematical events, how problems based on mathematics can be used to unfamiliar settings., The simplest way to learn Permutations and combinations and its algebra. Why was Theory of Binomial theorem and its coefficients invented? Learning of Permutation and combinations intuitively.

On the other hand, suggested Micro projects and brain storming questions create inquisitiveness and curiosity for the topics included in the unit.

RATIONALE

Combinatorial theory, is a foremost mathematics branch that has wide applications in many field generations of combinatorial sequences, such as permutations and combinations, has been studied broadly because of the fundamental nature and the importance in practical applications. Permutation and combination come into play all the time when we are thinking about things in the real world. Permutations are just a different way of looking at things in the world through Mathematics. Permutations and combinations are two of the most fundamental topics in Probability. They are very valuable in statistics, and therefore in all parts of mathematics as well as in real life hence becoming an important concept to be learnt by the students. Studying permutations and combinations is essential to comprehend the world around us because these two methods help us make better choices without overlooking any possibilities. With Permutations, you focus on lists of elements where their order matters. With Combinations on the other hand, the focus is on groups of elements where the order does not matter.

Binomial theorem has numerous applications owing to its ease of use and understandability, whether it is for making predictions in the economy or forecasting of weather. The wide range of its usage advocates that for learning higher mathematics or physics the command over this theorem is a necessity.

PRE-REQUISITE

- Basic skills for simplifying algebraic expressions.
- Expanding brackets.
- Factoring linear and quadratic expressions.
- Some experience in working with polynomials.
- Familiarity with the algebraic techniques.
- Substitution.

UNIT OUTCOMES

List of outcomes of this unit are as follows:

U5-O1: Calculate the number of permutations and combinations of ‘n’ objects taken ‘r’ at a time.

U5-O2: Apply the theory of permutations and combinations to solve the counting problems.

U5-O3: Compute Binomial coefficients by formula.

U5-O4: Use Binomial theorem to expand binomial expressions that are raised to positive integer powers.

U5-O5: Use Binomial Theorem to find approximation.

Unit Outcome	**Expected Mapping with Program Outcomes** *(1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation)*						
	CO-1	**CO-2**	**CO-3**	**CO-4**	**CO-5**	**CO-6**	**CO-7**
U5-O1	-	-	-	-	2	1	3
U5-O2	-	-	-	1	2	1	3
U5-O3	-	-	-	-	1	-	3
U5-O4	-	-	-	1	2	1	3
U5-O5	-	-	-	1	2	1	3

5.1 FUNDAMENTAL PRINCIPLE OF COUNTING

5.1.1 Principle of Multiplication

Let us consider the following problem. Mohan has 3 pants and 2 shirts. How many different pairs of a pant and a shirt, can he dress up with? There are 3 ways in which pant can be chosen, because there are 3 pants available. Similarly, a shirt can be chosen in 2 ways. For every choice of a pant, there are 2 choices of a shirt. Therefore, there are $3 \times 2 = 6$ pairs of a pant and a shirt. Let us name the three pants as P_1, P_2, P_3 and the two shirts as S_1, S_2. Then, these six possibilities can be illustrated in the Fig.

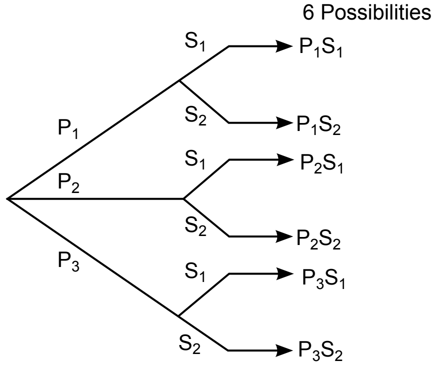

Fig. 5.1: Possibility Showing Multiplication-1

Let us consider another problem of the same type. Shabnam has 2 school bags, 3 tiffin boxes and 2 water bottles. In how many ways can she carry these items (choosing one each).

A school bag can be chosen in 2 different ways. After a school bag is chosen, a tiffin box can be chosen in 3 different ways. Hence, there are $2 \times 3 = 6$ pairs of school bag and a tiff inbox. For each of these pairs a water bottle can be chosen in 2 different ways. Hence, there are $6 \times 2 = 12$ different ways in which, Shabnam can carry these items to school. If we name the 2 school bags as B_1, B_2, the three tiffin boxes as T_1, T_2, T_3 and the two water bottles as W_1, W_2, these possibilities can be illustrated in the Fig.

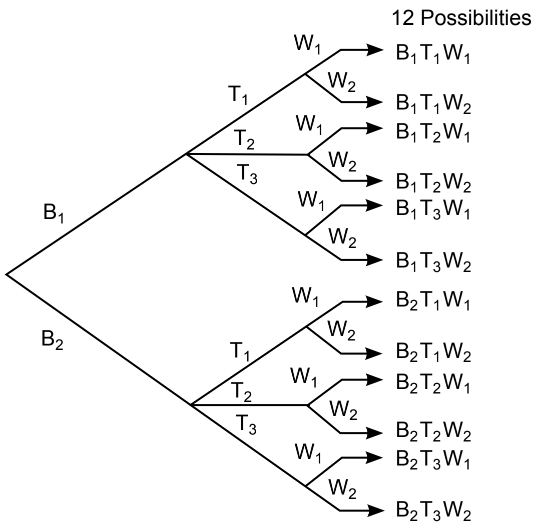

Fig. 5.2: Possibility Showing Multiplication-2

In fact, the problems of the above types are solved by applying the following principle known as the fundamental principle of counting, or, simply, the multiplication principle, which states that "If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrences of the events in the given order is m × n."

5.1.2 Principle of Addition

In first Example First event is to choose shirts, so it can be chosen as either Shirt S_1 or S_2 or S_3 here number of ways is 1+1+1=3 so here principle of addition is applied. Instead of wearing shirt suppose Mohan had 5 t-Shirt also then total number of ways of wearing shirts or T-Shirts would be 3+5=8.

Example 1: Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed.

Solution: There are as many words as there are ways of filling in 4 vacant places { }, { }, { }, { } by the 4 letters, keeping in mind that the repetition is not allowed. The first place can be filled in 4 different ways by anyone of the 4 letters R, O, S, E. Following which, the second place can be filled in by anyone of the remaining 3 letters in 3 different ways, following which the third place can be filled in 2 different ways; following which, the fourth place can be filled in 1 way. Thus, the number of ways in which the 4 places can be filled, by the multiplication principle, is $4 \times 3 \times 2 \times 1 = 24$. Hence, the required number of words is 24.

Note: If the repetition of the letters was allowed, how many words can be formed? One can easily understand that each of the 4 vacant places can be filled in succession in 4 different ways. Hence, the required number of words = $4 \times 4 \times 4 \times 4 = 256$.

Example 2: How many 2 digits even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

Solution There will be as many ways as there are ways of filling 2 vacant places in succession by the five given digits. Here, in this case, we start filling in unit's place, because the options for this place are 2 and 4 only and this can be done in 2 ways; following which the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated. Therefore, by the multiplication principle, the required number of two digits even numbers are 2×5, i.e., 10.

5.2 PERMUTATIONS

In Example 1 of the previous Section, we are actually counting the different possible arrangements of the letters such as ROSE, REOS, ..., etc. Here, in this list, each arrangement is different from other. In other words, the order of writing the letters is important. Each arrangement is called a permutation of 4 different letters taken all at a time. Now, if we have to determine the number of 3-letter words, with or without meaning, which can be formed out of the letters of the word NUMBER, where the repetition of the letters is not allowed, we need to count the arrangements NUM, NMU, MUN, NUB, ..., etc. Here, we are counting the permutations of 6 different letters taken 3 at a time. The required number of words = $6 \times 5 \times 4 = 120$ (by using multiplication principle).

If the repetition of the letters was allowed, the required number of words would be $6 \times 6 \times 6 = 216$.

5.2.1 Permutations when all the Objects are Distinct

The number of permutations of n different objects taken r at a time, where $0 < r \leq n$ and the objects do not repeat is n (n– 1) (n– 2). . .($n - r + 1$), which is denoted by ${}^{n}P_{r}$.

This expression for ^nP_r is cumbersome and we need a notation which will help to reduce the size of this expression. The symbol $n!$ (read as factorial n or n factorial) comes to our rescue. In the following text we will learn what actually $n!$ means.

5.2.2 Factorial Notation

Factorial notation the notation $n!$ represents the product of first n natural numbers, i.e., the product $1 \times 2 \times 3 \times \ldots \times (n-1) \times n$ is denoted as $n!$ We read this symbol as 'n factorial'.

Thus, $1 \times 2 \times 3 \times 4 \ldots \times (n-1) \times n = n!$

$1 = 1!$

$1 \times 2 = 2!$

$1 \times 2 \times 3 = 3!$

$1 \times 2 \times 3 \times 4 = 4!$ and so on.

We define $0! = 1$

We can write $5! = 5 \times 4! = 5 \times 4 \times 3! = 5 \times 4 \times 3 \times 2!$

$= 5 \times 4 \times 3 \times 2 \times 1!$

Clearly, for a natural number n

$n! = n(n-1)!$

$= n(n-1)(n-2)!$ [provided $(n \geq 2)$]

$= n(n-1)(n-2)(n-3)!$ [provided $(n \geq 3)$]

and so on.

Example 3: Evaluate

(i) $5!$ (ii) $7!$ (iii) $7! - 5!$

Solution:

(i) $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

(ii) $7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$

(iii) $7! - 5! = 5040 - 120 = 4920.$

Example 4: Find the value of the following.

(i) $^5P_2$ (ii) $^4P_4$ (iii) $P(6,0)$

Solution:

(i) $$^5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \times 4 \times 3!}{3!} = 20$$

(ii) $$^4P_4 = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 4! = 24 \quad \{\because 0! = 1$$

(iii) $$P(6,0) = \frac{6!}{(6-0)!} = \frac{6!}{6!} = 1$$

Example 5: If $\frac{1}{8!}+\frac{1}{9!}=\frac{x}{10!}$ Find x

Solution: We have

$$\frac{1}{8!}+\frac{1}{9\times 8!}=\frac{x}{10\times 9\times 8!}\Rightarrow 1+\frac{1}{9}=\frac{x}{10\times 9}\,or\,\frac{10}{9}=\frac{x}{90}\text{, So } x=100$$

Example 6: Prove that $(n-1)!+(n+1)!=(n^2+n+1)(n-1)!$

Solution:

$$LHS=(n-1)!+(n+1)!=(n-1)!+(n+1)(n)(n-1)!=(n-1)!(1+n^2+n)=(1+n^2+n)(n-1)!=RHS$$

Example 7: Prove that

$$2\cdot 4\cdot 6\cdots\cdots(2n-2)=\frac{2(n!)}{n}$$

Solution:

$$LHS=2\cdot 4\cdot 6\cdots\cdots(2n-2)=2\left[1\cdot 2\cdot 3\cdots\cdots(n-1)\right]=2(n-1)!=2\left[\cdot\frac{(n-1)!n}{n}\right]=\frac{2(n!)}{n}=RHS$$

Example 8: Prove that $\frac{n!}{r!(n-r)!}+\frac{n!}{(r-1)!(n-r+1)!}=\frac{(n+1)!}{r!(n-r+1)!}$

Solution:

$$LHS=\frac{n!}{r!(n-r)!}+\frac{n!}{(r-1)!(n-r+1)!}$$

$$=\frac{n!}{r!(n-r)!}\cdot\frac{(n-r+1)}{(n-r+1)}+\frac{n!}{(r-1)!(n-r+1)!}\cdot\frac{r}{r}$$

$$=\frac{(n-r+1)n!}{r!(n-r+1)!}+\frac{rn!}{(r)!(n-r+1)!}=\frac{(n-r+1+r)n!}{r!(n-r+1)!}$$

$$=\frac{(n+1)n!}{r!(n-r+1)!}=\frac{(n+1)!}{r!(n-r+1)!}$$

$$=RHS$$

5.2.3 Permutation Under Various Case

Case 1. Out of n objects r to be arranged in a line so total number of arrangements can be obtained by formula

$$^{n}P_{r}\text{ or }\mathrm{P}(\mathrm{n},\mathrm{r})=\frac{n!}{(n-r)!}\ ;\ n\geq r\geq 0$$

Case 2. The number of permutations of n different objects taken r at a time, where repetition is allowed, is n^r

Case 3. Permutations when all the objects are not distinct objects: Suppose we have to find the number of ways of rearranging the letters of the word ROOT. In this case, the letters of the word are not all different. There are 2 Os, which are of the same kind. Let us treat, temporarily, the 2 Os as different, say, O_1 and O_2. The number of permutations of 4-different letters, in this case, taken all at a time is 4! Consider one of these permutations say, RO_1O_2T. Corresponding to this permutation, we have 2! permutations RO_1O_2T and RO_2O_1T which will be exactly the same permutation if O_1 and O_2 are not treated as different, i.e., if O_1 and O_2 are the same O at both places.

Therefore, the required number of permutations $\dfrac{4!}{2!} = 4 \times 3 = 12$

Permutations when O_1, O_2 are different	Permutations when O_1, O_2 are the same O.
RO_1O_2T, RO_2O_1T	R O O T
TO_1O_2R, TO_2O_1R	T O O R
RO_1T_2O, RO_2T_1O	R O T O
TO_1RO_2, TO_2RO_1	T O R O
RTO_1O_2, RTO_2O_1	R T O O
TRO_1O_2, TRO_2O_1	T R O O
O_1O_2RT, O_2O_1TR	O O R T
O_1RO_2T, O_2RO_1T	O R O T
O_1TO_2R, O_2TO_1R	O T O R
O_1RTO_2, O_2RTO_1	O R T O
O_1TRO_2, O_2TRO_1	O T R O
O_1O_2TR, O_2O_1TR	O O T R

Fig. 5.3: Representation of Permutation

Example 9: How many 4-digit numbers can be formed by using the digits 1 to 9 if repetition of digits is not allowed?

Solution: Here order matters for example 1234 and 1324 are two different numbers. Therefore, there will be as many 4-digit numbers as there are permutations of 9 different digits taken 4 at a time.

Therefore, the required 4-digit numbers $^9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = 9\times8\times7\times6 = 3024$

Example 10: 12 STUDENTS PASSED IN Mid-semester examination with distinct marks. In how many ways can the first three prizes be won?

Solution: Here we have 12 students and 3 prizes. All 12 students obtained distinct marks so no one can win more than one prize and not more than one student win same prize.

Therefore, the required number of ways = $^{12}P_3 = \frac{12!}{(12-3)!} = \frac{12!}{9!} = 12\times11\times10 = 1320$

Example 11: Find the number of all 6-digit numbers with distinct digits.

Solution: We know that there are 10 digits. i.e. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Here we have to arrange 6 of these 10 in a row to form 6-digits number.

This can be done in P (10,6) ways.

$$^{10}P_6 = \frac{10!}{(10-6)!} = \frac{10!}{4!} = 10\times9\times8\times7\times6\times5 = 151200$$

But the numbers whose extreme left place is 0(zero) are not a 6-digit numbers.

There are P(9,5) numbers of such type

$$i.e.\ ^9P_5 = \frac{9!}{(9-5)!} = \frac{9!}{4!} = 9\times8\times7\times6\times5 = 15120$$

Hence the required number of 6-digit numbers with distinct digits are

= 151200 – 15120 = 136080

Case 4. May be generalised for the number of permutations of n objects, where $p1$ objects are of one kind, $p2$ are of second kind, ..., pk are of kth kind and the rest, if any, are of different kind is

$$\frac{n!}{p_1!p_2!\ldots p_k!}$$

Example 12: Find the number of permutations of the letters of the word ALLAHABAD.

Solution: Here, there are 9 objects (letters) of which there are 4A's, 2 L's and rest are all different.

Therefore, the required number of arrangements = $\frac{9!}{4!2!} = \frac{5\times6\times7\times8\times9}{2} = 7560$

Example 13: Find the value of n such that $^nP_5 = 42\left(^nP_3\right), n > 4$

Solution: Given that

$$^nP_5 = 42\left(^nP_3\right), n > 4$$

or $n(n-1)(n-2)(n-3)(n-4) = 42n(n-1)(n-2)$

Since $n > 4$ $so\, n(n-1)(n-2) \neq 0$

Therefore, by dividing both sides by $n\ (n - 1)\ (n - 2)$, we get

$(n-3)(n-4) = 42$

$n^2 - 7n + 12 - 42 = 0p1$

$n^2 - 7n - 30 = 0$

$n^2 - 10n + 3n - 30 = 0$

$(n-10)(n+3) = 0$

Or $n - 10 = 0$ or $n + 3 = 0$

Or $n = 10$ or $n = -3$

Therefore, n=10 as n>4.

Permutation & Combination

5.3 COMBINATIONS

Let us now assume that there is a group of 3 lawn tennis players X, Y, Z. A team consisting of 2 players is to be formed. In how many ways can we do so? Is the team of X and Y different from the team of Y and X? Here, order is not important. In fact, there are only 3 possible ways in which the team could be constructed.

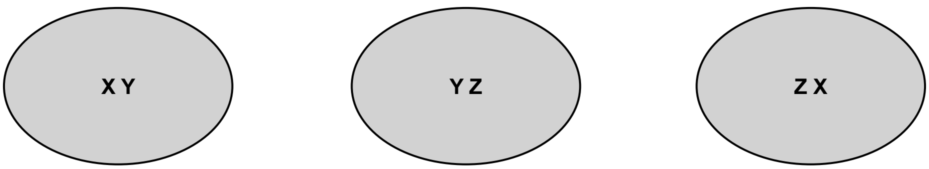

Fig. 5.4: Representation of Combination

These are XY, YZ and ZX. Here, each selection is called a *combination of 3 different objects taken 2 at a time*. ***In a combination, the order is not important***

Now consider some more illustrations:

1. Twelve persons meet in a room and each shakes hand with all the others. How do we determine the number of handshakes? X shaking hands with Y and Y with X will not be two different handshakes. Here, order is not important. There will be as many handshakes as there are combinations of 12 different things taken 2 at a time.
2. Seven points lie on a circle. How many chords can be drawn by joining these points pair wise? There will be as many chords as there are combinations of 7 different things taken 2 at a time.

Now, we obtain the formula for finding the number of combinations of *n* different objects taken *r* at a time, denoted by n_{C_r} which is defined as

$${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

In particular,

1. $If\ r = n, {}^{n}C_{n} = \frac{n!}{n!(n-n)!} = 1$

2. $If\ r = 0, {}^{n}C_{0} = \frac{n!}{0!(n-0)!} = 1$

3. ${}^{n}C_{n-r} = {}^{n}C_{r}$ i.e., selecting r objects out of n objects is same as rejecting $(n - r)$ objects.

4. $If\ {}^{n}C_{a} = {}^{n}C_{b}$, then $a = b \text{ or } n = a + b$

5. ${}^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r}$

Example 14: If ${}^{n}C_{9} = {}^{n}C_{8}$ then evaluate ${}^{n}C_{17}$

Solution:

Since ${}^{n}C_{a} = {}^{n}C_{b}$ then $a = b or n = a + b$

So $n = 9 + 8 = 17$

Now ${}^{n}C_{17} = {}^{17}C_{17} = 1$

Example 15: Find the value of the following.

(i) ${}^{7}C_{3}$ (ii) ${}^{5}C_{5}$ (iii) ${}^{13}C_{0}$

Solution:

(i) $C_{3}^{7} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!(4)!} = \frac{7\times6\times5\times4!}{3\times2\times1\times4!} = 35$

(ii) ${}^{5}C_{5} = \frac{5!}{5!(5-5)1} = \frac{5!}{(5!)(0!)} = \frac{5!}{5!} = 1$

(iii) ${}^{13}C_{0} = \frac{13!}{0!(13-0)!} = \frac{13!}{(0!)(13!)} = \frac{13!}{13!} = 1$

Example 16: A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

Solution: Here, order does not matter. Therefore, we need to count combinations. There will be as many committees as there are combinations of 5 different persons taken 3 at a time. Hence, the required number of ways = ${}^{5}C_{3} = \frac{5!}{3!(5-3)!} = \frac{5\times4\times3!}{3!\times2\times1} = 10$

Now, 1 man can be selected from 2 men in ${}^{2}C_{1}$ ways and 2 women can be selected from 3 women in ${}^{3}C_{2}$ ways. Therefore, the required number of committees = ${}^{2}C_{1} \times {}^{3}C_{2} = 6$

Permutation & Combination-2

5.4 BINOMIAL EXPRESSION

An algebraic expression consisting of two terms with +*ve* or – *ve* sign between them is called a binomial expression.

For example: $(a+b), (2x-3y), \left(\frac{p}{x^2}-\frac{q}{x^4}\right), \left(\frac{1}{x}+\frac{4}{y^3}\right)$ etc.

5.4.1 Binomial Theorem for Positive Integral Index

The rule by which any power of binomial can be expanded is called the binomial theorem.

If n is a positive integer and $x,\ y \in C$ then

$$\left(x-y\right)^n = {}^nC_0x^{n-0}y^0 - {}^nC_1x^{n-1}y^1 + {}^nC_2x^{n-2}y^2 - \ldots\ldots + {}^nC_rx^{n-r}y^r + \ldots\ldots + {}^nC_{n-1}xy^{n-1} + {}^nC_nx^0y^n$$

$$i.e.,\ (x+y)^n = \sum_{r=0}^{n} {}^nC_r.x^{n-r}.y^r \qquad \ldots.(i)$$

Here ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots {}^nC_n$ are called binomial coefficients and $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots\ldots$ for $0 \le r \le n$.

Some important expansions:

(1) Replacing y by – y in (i), we get,

$$\left(x-y\right)^n = {}^nC_0x^{n-0}y^0 - {}^nC_1x^{n-1}y^1 + {}^nC_2x^{n-2}y^2 - \ldots\ldots + (-1)^r\ {}^nC_rx^{n-r}y^r + \ldots. + (-1)^n\ {}^nC_nx^0y^n$$

$$i.e.,\ (x-y)^n = \sum_{r=0}^{n} (-1)^r\ {}^nC_rx^{n-r}y^r$$

The terms in the expansion of $(x-y)^n$ are alternatively positive and negative, the last term is positive or negative according as n is even or odd.

(2) Replacing x by 1 and y by x in equation (i) we get,

$$(1+x)^n = {}^nC_0x^0 + {}^nC_1x^1 + {}^nC_2x^2 + \ldots\ldots + {}^nC_rx^r + \ldots\ldots + {}^nC_nx^n$$

$$i.e.,\ (1+x)^n = \sum_{r=0}^{n} {}^nC_rx^r$$

This is expansion of $(1+x)^n$ in ascending power of x.

(3) Replacing x by 1 and y by – x in (i) we get,

$$(1-x)^n = {}^nC_0x^0 - {}^nC_1x^1 + {}^nC_2x^2 - \ldots\ldots + \left(-1\right)^r\ {}^nC_rx^r + \ldots\ldots + \left(-1\right)^n\ {}^nC_nx^n$$

$$i.e.,\ (1-x)^n = \sum_{r=0}^{n} (-1)^r\ {}^nC_rx^r$$

(4) $(x+y)^n+(x-y)^n = 2[{}^nC_0x^ny^0+{}^nC_2x^{n-2}y^2+{}^nC_4x^{n-4}y^4+.......]$ and

$(x+y)^n-(x-y)^n=2[{}^nC_1x^{n-1}y^1+{}^nC_3x^{n-3}y^3+{}^nC_5x^{n-5}y^5+...]$

(5) The coefficient of $(r+1)^{th}$ term in the expansion of $(1+x)^n$ is ${}^nC_r$.

(6) The coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$.

General term

The general term of the expansion is $(r+1)^{th}$ term usually denoted by T_{r+1} and $T_{r+1}={}^nC_rx^{n-r}y^r$

- In the binomial expansion of $(x-y)^n, T_{r+1}=(-1)^r\,{}^nC_rx^{n-r}y^r$
- In the binomial expansion of $(1+x)^n, T_{r+1}={}^nC_rx^r$
- In the binomial expansion of $(1-x)^n, T_{r+1}=(-1)^r\,{}^nC_rx^r$

5.4.2 Binomial Theorem for any Index

Statement:

$$(1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)}{3!}x^3+....+\frac{n(n-1)......(n-r+1)}{r!}x^r+...\text{terms up to }\infty$$

when n is a negative integer or a fraction, where $-1<x<1$, otherwise expansion will not be possible.

If first term is not 1, then make first term unity in the following way, $(x+y)^n=x^n\left[1+\frac{y}{x}\right]^n$, if $\left|\frac{y}{x}\right|<1$.

General term:

$$T_{r+1}=\frac{n(n-1)(n-2)......(n-r+1)}{r!}x^r$$

Some important expansions:

1. $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+......+\frac{n(n-1)(n-2)......(n-r+1)}{r!}x^r+......$
2. $(1-x)^n=1-nx+\frac{n(n-1)}{2!}x^2-.......+\frac{n(n-1)(n-2).....(n-r+1)}{r!}(-x)^r+.......$
3. $(1-x)^{-n}=1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+.....+\frac{n(n+1)......(n+r-1)}{r!}x^r+.....$

4. $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots + \frac{n(n+1)\ldots\ldots(n+r-1)}{r!}(-x)^r + \ldots\ldots$

5. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\infty$

6. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\infty$

7. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\infty$

8. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\infty$

9. $(1+x)^{-3} = 1 - 3x + 6x^2 - \ldots\ldots\ldots\ldots\infty$

10. $(1-x)^{-3} = 1 + 3x + 6x^2 + \ldots\ldots\ldots\ldots\ldots\infty$

5.4.3 Problems on Approximation by the Binomial Theorem

We have $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots$.

If x is small compared with 1, we find that the values of $x^2, x^3, x^4, \ldots\ldots$ become smaller and smaller.

$\therefore$ The terms in the above expansion become smaller and smaller. If x is very small compared with 1, we might take 1 as a first approximation to the value of $(1+x)^n$ or 1 + nx as a second approximation.

Example 17: Expand $\left(x^2 + \frac{3}{x}\right)^4 ; x \neq 0$

Solution: We know that the binomial theorem is,

$$(a+b)^n = {}^nC_0(a)^n(b)^0 + {}^nC_1(a)^{n-1}(b)^1 + {}^nC_2(a)^{n-2}(b)^2 + \ldots\ldots\ldots + {}^nC_n(a)^0(b)^n$$

Now by comparing with binomial theorem we have,

$a = x^2, b = \frac{3}{x}$ and $n = 4$

$$\left(x^2 + \frac{3}{x}\right)^4 = {}^4C_0\left(x^2\right)^4\left(\frac{3}{x}\right)^0 + {}^4C_1\left(x^2\right)^3\left(\frac{3}{x}\right)^1 + {}^4C_2\left(x^2\right)^2\left(\frac{3}{x}\right)^2$$
$$+ {}^4C_3\left(x^2\right)^1\left(\frac{3}{x}\right)^3 + {}^4C_4\left(x^2\right)^0\left(\frac{3}{x}\right)^4$$
$$= x^8 + 4x^6.\frac{3}{x} + 6x^4.\frac{9}{x^2} + 4x^2.\frac{27}{x^3} + \frac{81}{x^4} = x^8 + 15x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}$$

Example 18: Expand the following using Binomial Theorem.

(i) $\left(-3x+\dfrac{4}{x^2}\right)^4$ (ii) $\left(\dfrac{2x^2}{3}-\dfrac{3}{x^2}\right)^6$

(iii) $(2x+3y)^5$ (iv) $(1+x+x^2)^5$

Solution:

(i) $\left(-3x+\dfrac{4}{x^2}\right)^4$

We know that the binomial theorem is,

$$(a+b)^n = {}^nC_0(a)^n(b)^0 + {}^nC_1(a)^{n-1}(b)^1 + {}^nC_2(a)^{n-2}(b)^2 + \dots\dots\dots + {}^nC_n(a)^0(b)^n$$

Now by comparing with binomial theorem we have,

$a=-3x, b=\dfrac{4}{x}$ and $n=4$

$$\therefore \left(-3x+\frac{4}{x^2}\right)^4 = {}^4C_0(-3x)^4\left(\frac{4}{x^2}\right)^0 + {}^4C_1(-3x)^{4-1}\left(\frac{4}{x^2}\right)^1 + {}^4C_2(-3x)^{4-2}\left(\frac{4}{x^2}\right)^2 + {}^4C_3(-3x)^{4-3}\left(\frac{4}{x^2}\right)^3$$
$$+ {}^4C_4(-3x)^{4-4}\left(\frac{4}{x^2}\right)^4$$

$$=1\cdot 81x^4\cdot 1+4\cdot(-27)x^3\cdot\frac{4}{x^2}+6\cdot 9x^2\cdot\frac{16}{x^4}+4\cdot(-3)x\cdot\frac{64}{x^6}+1\cdot 1\cdot\frac{256}{x^8}$$

$$=81x^4-432x+\frac{864}{x^2}-\frac{768}{x^5}+\frac{256}{x^8}$$

(ii) $\left(\dfrac{2x^2}{3}-\dfrac{3}{x^2}\right)^6$

We know that the binomial theorem is,

$$(a+b)^n = {}^nC_0(a)^n(b)^0 + {}^nC_1(a)^{n-1}(b)^1 + {}^nC_2(a)^{n-2}(b)^2 + \dots\dots\dots + {}^nC_n(a)^0(b)^n$$

Now by comparing with binomial theorem we have,

$a=\dfrac{2x^2}{3}, b=-\dfrac{3}{x^2}$ and $n=6$

$$\therefore \left(\frac{2x^2}{3}-\frac{3}{x^2}\right)^6 = {}^6C_0\left(\frac{2x^2}{3}\right)^6\left(-\frac{3}{x^2}\right)^0 + {}^6C_1\left(\frac{2x^2}{3}\right)^5\left(-\frac{3}{x^2}\right)^1 + {}^6C_2\left(\frac{2x^2}{3}\right)^4\left(-\frac{3}{x^2}\right)^2 + {}^6C_3\left(\frac{2x^2}{3}\right)^3\left(-\frac{3}{x^2}\right)^3$$
$$+ {}^6C_4\left(\frac{2x^2}{3}\right)^2\left(-\frac{3}{x^2}\right)^4 + {}^6C_5\left(\frac{2x^2}{3}\right)^1\left(-\frac{3}{x^2}\right)^5 + {}^6C_6\left(\frac{2x^2}{3}\right)^0\left(-\frac{3}{x^2}\right)^6$$

$$= 1\cdot\left(\frac{64x^{12}}{729}\right)\cdot 1 + 6\left(\frac{32x^{10}}{243}\right)\left(-\frac{3}{x^2}\right) + 15\left(\frac{16x^2}{81}\right)\left(\frac{9}{x^4}\right) + 20\left(\frac{8x^6}{27}\right)\left(-\frac{27}{x^6}\right)$$
$$+15\left(\frac{4x^4}{9}\right)\left(\frac{81}{x^8}\right) + 6\left(\frac{2x^2}{3}\right)\left(-\frac{243}{x^{10}}\right) + 1\cdot 1\cdot\left(\frac{729}{x^{12}}\right)$$

$$= \frac{64x^{12}}{729} - \frac{64x^8}{27} + \frac{80x^4}{3} - 160 + \frac{540}{x^4} - \frac{972}{x^8} + \frac{729}{x^{12}}$$

(iii) $(2x+3y)^5$

We know that the binomial theorem is,

$$(a+b)^n = {}^nC_0(a)^n(b)^0 + {}^nC_1(a)^{n-1}(b)^1 + {}^nC_2(a)^{n-2}(b)^2 + \dots\dots\dots + {}^nC_n(a)^0(b)^n$$

Now by comparing with binomial theorem we have,

$a = 2x$, $b = 3y$ and $n = 5$

$$\therefore (2x+3y)^5 = {}^5C_0(2x)^5(3y)^0 + {}^5C_1(2x)^4(3y)^1 + {}^5C_2(2x)^3(3y)^2 + {}^5C_3(2x)^2(3y)^3$$
$$+ {}^5C_4(2x)^1(3y)^4 + {}^5C_5(2x)^0(3y)^5$$

$$\therefore (2x+3y)^5 = 1\cdot(32x^5)\cdot 1 + 5(16x^4)(3y) + 10(8x^3)(9y^2) + 10(4x^2)(27y^3) + 5(2x)(81y^4) + 1\cdot 1\cdot(243y^5)$$

$$\therefore (2x+3y)^5 = 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$$

(iv) $(1+x+x^2)^5$

$$Let\ 1+x = y\ \therefore (1+x+x^2)^5 = (y+x^2)^5$$

$$(y+x^2)^5 = \therefore (y+x^2)^5 = {}^5C_0(y)^5(x^2)^0 + {}^5C_1(y)^4(x^2)^1 + {}^5C_2(y)^3(x^2)^2 + {}^5C_3(y)^2(x^2)^3$$
$$+ {}^5C_4(y)^1(x^2)^4 + {}^5C_5(y)^0(x^2)^5$$

$$\therefore (y+x^2)^5 = 1\cdot(y)^5\cdot 1 + 5(y)^4(x^2)^1 + 10(y)^3(x^2)^2 + 10(y)^2(x^2)^3 + 5(y)^1(x^2)^4 + 1\cdot 1\cdot(x^2)^5$$

$$\therefore (y+x^2)^5 = y^5 + 5y^4x^2 + 10y^3x^4 + 10y^2x^6 + 5yx^8 + x^{10}$$

$$\therefore (1+x+x^2)^5 = (1+x)^5 + 5(1+x)^4x^2 + 10(1+x)^3x^4 + 10(1+x)^2x^6 + 5(1+x)x^8 + x^{10}$$

Using

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \dots\dots\dots + {}^nC_nx^{n-1}$$

We have

$$\left(1+x\right)^5 = 1+5x+10x^2+10x^3+5x^4+x^5$$

$$\left(1+x\right)^4 = 1+4x+6x^2+4x^3+x^4$$

$\left(1+x\right)^3 = 1+3x+3x^2+x^3$ and $\left(1+x\right)^2 = 1+2x+x^2$

$$\therefore \left(1+x+x^2\right)^5 = \left(1+5x+10x^2+10x^3+5x^4+x^5\right)+5\left(1+4x+6x^2+4x^3+x^4\right)x^2$$
$$+10\left(1+3x+3x^2+x^3\right)x^4+10\left(1+2x+x^2\right)x^6+5\left(1+x\right)x^8+x^{10}$$

$$\therefore \left(1+x+x^2\right)^5 = 1+5x+10x^2+10x^3+5x^4+x^5+5x^2+20x^3+30x^4+20x^5+5x^6$$
$$+10x^4+30x^5+30x^6+10x^7+10x^6+20x^7+10x^8+5x^8+5x^9+x^{10}$$

$$\therefore \left(1+x+x^2\right)^5 = 1+5x+15x^2+30x^3+45x^4+51x^5+45x^6+30x^7+15x^8+5x^9+x^{10}$$

Example 19: Do as directed.

(i) Find 5^{th} term in $\left(x^2+2y\right)^8$

(ii) Find the middle term in $\left(\dfrac{2x^2}{3}+\dfrac{3}{2x^2}\right)^{12}$

(iii) Find the term independent of y in the expansion of $\left(\dfrac{y^2}{3}-\dfrac{4}{y^2}\right)^6$

Solution:

(i) Find 5^{th} term in $\left(x^2+2y\right)^8$

The $\left(r+1\right)^{th}$ term of $\left(a+b\right)^n$ is, $T_{r+1} = {}^nC_r\left(a\right)^{n-r}\left(b\right)^r$

Here $a=x^2, b=2y$ and $n=8$

For finding 5^{th} we have to take r=4

$$\therefore T_{r+1} = C_4^8\left(x^2\right)^{8-4}\left(2y\right)^4 = \frac{8\times7\times6\times5\times4!}{4\times3\times2\times1\times4!}\times x^8\times16y^4 = 224x^8y^4$$

(ii) Find the middle term in $\left(\dfrac{2x^2}{3}+\dfrac{3}{2x^2}\right)^{12}$

Number of terms in the expansion is 12 + 1 = 13

Therefore, only one middle term.

Here $\left(\frac{12}{2}+1\right)^{th}=7^{th}$ term is the middle term.

$$a=\frac{2x^2}{3},b=\frac{3}{2x^2},n=12\,\text{and}\,r=6$$

$$T_{6+1}=C_6^{12}\left(\frac{2x^2}{3}\right)^{12-6}\left(\frac{3}{2x^2}\right)^6=\frac{12\times11\times10\times9\times8\times7\times6!}{6\times5\times4\times3\times2\times1\times6!}\times\frac{(2)^6x^{12}}{(3)^6}\times\frac{(3)^6}{(2)^6x^{12}}=924$$

(iii) Find the term independent of y in the expansion of $\left(\frac{y^2}{3}-\frac{4}{y^2}\right)^6$

Let T_{r+1} be the term independent of y in theexpansion of $\left(\frac{y^2}{3}-\frac{4}{y^2}\right)^6$

$$\therefore T_{r+1}={}^6C_r\left(\frac{y^2}{3}\right)^{6-r}\left(-\frac{4}{y^2}\right)^r=\frac{6!}{r!(6-r)!}\times\frac{y^{12-2r}}{(3)^{6-r}}\times\frac{(-4)^r}{y^{2r}}=\frac{6!}{r!(6-r)!}\times\frac{(-4)^r}{(3)^{6-r}}\times y^{12-4r}$$

Now, the required term is independent of y means the term having power of y is zero.

$\therefore 12-4r=0$ $\qquad\qquad$ $\therefore r=3$

$$\therefore T_{r+1}=\frac{6!}{3!(6-3)!}\times\frac{(-4)^3}{(3)^{6-3}}\times y^0=-\frac{1280}{27}$$

Example 20: Simplify $(x+y)^4+(x-y)^4$ and hence evaluate $\left(\sqrt{2}+1\right)^4+\left(\sqrt{2}-1\right)^4$

Solution:

$$(x+y)^4+(x-y)^4=\left\{\begin{array}{l}\left[{}^nC_0(x)^4(y)^0+{}^4C_1(x)^3(y)^1+{}^4C_2(x)^2(y)^2+{}^4C_3(x)^1(y)^3+{}^4C_4(x)^0(y)^4\right]\\+\left[{}^4C_0(x)^4(-y)^0+{}^4C_1(x)^3(-y)^1+{}^4C_2(x)^2(-y)^2+{}^4C_3(x)^1(-y)^3+{}^4C_4(x)^0(-y)^4\right]\end{array}\right\}$$

$$(x+y)^4+(x-y)^4=\left\{\begin{array}{l}\left[1\cdot(x)^4\cdot1+4(x)^3(y)^1+6(x)^2(y)^2+4(x)^1(y)^3+1\cdot1\cdot y^4\right]\\+\left[1\cdot(x)^4\cdot1-4(x)^3(y)^1+6(x)^2(y)^2-4(x)^1(y)^3+1\cdot1\cdot y^4\right]\end{array}\right\}$$

$$(x+y)^4+(x-y)^4=2\left(x^4+6x^2y^2+y^4\right)$$

$$\therefore\left(\sqrt{2}+1\right)^4+\left(\sqrt{2}-1\right)^4=2\left(\left(\sqrt{2}\right)^4+6\left(\sqrt{2}\right)^2(1)^2+(1)^4\right)=2(4+12+1)=34$$

Example 21: Find which is larger $99^{50}+100^{50}$ or 101^{50}

Solution: We have

$$101^{50}=(100+1)^{50}=100^{50}+50.100^{49}+\frac{50.49}{2.1}100^{48}+.... \qquad .....(i)$$

and $$99^{50}=(100-1)^{50}=100^{50}-50.100^{49}+\frac{50.49}{2.1}100^{48}-.. \qquad ....(ii)$$

Subtracting (ii) from (i), we get

$$101^{50}-99^{50}=2\cdot 50\cdot 100^{49}+2\cdot\frac{50\cdot 49\cdot 48}{1\cdot 2\cdot 3}\cdot 100^{47}=100^{50}+2\frac{50.49.48}{1.2.3}100^{47}>100^{50}$$

Hence $101^{50} > 100^{50}+99^{50}$.

Example 22: Show that $(1+x)^n-nx-1$ divisible by x^2 (where $n\in N$)

Solution:

$$(1+x)^n=1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+....$$

$$\therefore\ (1+x)^n-nx-1=x^2\left[\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}x+....\right]$$

From above it is clear that $(1+x)^n-nx-1$ is divisible by x^2.

Example 23: Find the 6^{th} term in expansion of $\left(2x^2-\frac{1}{3x^2}\right)^{10}$

Solution:

Applying $T_{r+1}={}^nC_r x^{n-r}a^r$ for $(x+a)^n$

Hence $T_6={}^{10}C_5(2x^2)^5\left(-\frac{1}{3x^2}\right)^5$

$$=-\frac{10!}{5!5!}32\times\frac{1}{243}=-\frac{896}{27}$$

Example 24: Using Binomial theorem, evaluate each of the following.

(i) $(96)^4$ (ii) $(101)^3$

Solution:

(i) $(96)^4=(100-4)^4$

$$={}^4C_0(100)^4(-4)^0+{}^4C_1(100)^3(-4)^1+{}^4C_2(100)^2(-4)^2$$
$$+{}^4C_3(100)^1(-4)^3+{}^4C_4(100)^0(-4)^4$$

$$= 1\cdot(100)^4\cdot 1 + 4\cdot(100)^3(-4)^1 + 6\cdot(100)^2(-4)^2 + 4\cdot(100)^1(-4)^3 + 1\cdot 1\cdot(-4)^4$$

$$= (100)^4 - 16(100)^3 + 96(100)^2 - 25600 + 256$$

$$= 100000000 - 16000000 + 960000 - 25600 + 256 = 84934656$$

(ii) $(101)^3 = (100+1)^3$

$$= {}^3C_0(100)^3(1)^0 + {}^3C_1(100)^2(1)^1 + {}^3C_2(100)^1(1)^2 + {}^3C_3(100)^0(1)^3$$

$$= 1\cdot(100)^3\cdot 1 + 3\cdot(100)^2(1)^1 + 3\cdot(100)^1(1)^2 + 1\cdot(100)^0(1)^3$$

$$= 1000000 + 30000 + 300 + 1 = 1030301$$

APPLICATIONS (REAL LIFE / INDUSTRIAL)

Simulation

Example 1: Permutations and combinations can be employed for simulations in many areas. Permutations representing various genotype-phenotype associations are employed in genetics simulations. Others are that employ permutations and combinations for simulations include networks, cryptography, databases and Operation Research.

SIM Card

Example 2: The amount of variance number for SIM Card can be computed as an application of theory of permutation.

Security Code

Example 3: The theory of permutation can be applied to the science of encryption or security code (password). Permutations are frequently used in communication networks and parallel and distributed systems.

Cryptography and Network Security

Example 4: Routing different permutations on a network for performance estimation is a common problem in the field of Cryptography and Network Security. Many communication networks require secure transfer of information, which drives development in cryptography and network security.

Forecast services

Example 5: Binomial theorem can be used in the prediction of upcoming disasters; this Theorem can also be used in predicting and analyzing weather patterns.

Finance:

Example 6: Binomial theorem is helpful to assist with the calculation of interest that is received after a span of several years at a certain interest rate on a sum of money.

Higher Mathematics

Example 7: Binomial Theorem is used in finding roots of equations in higher powers. It is also used in proving many significant equations in physics and mathematics.

Statistics and Probability

Example-8: The binomial theorem has variety of the applications in Statistics and Probability analyses of findings of results obtained are broadly used in our economy.

Architecture

Example 9: In order to estimate cost in engineering projects, Architecture utilize the applications of Binomial Theorem, they also apply to design of structure.

Internet protocol

Example 10: Binomial theorem has powerful application in Internet of Things (IoT). Another application can be found in variable subnetting.

UNIT SUMMARY

In this unit the first section is devoted to Permutations and Combinations, along with Value of ${}^nP_r$ and ${}^nC_r$. and its applications to real life problems. The study of permutations and combinations encourages and requires conception, algebraic ease, and an attention to precise calculations. Subsequent sections deal with Binomial theorem for positive integral index and for any index. Algebraic properties are discussed. New examples designed to reinforce the main concepts along with the applications of Permutations and Combinations. Some open questions are also advised these Verbal questions will help for assessing conceptual understanding of key terms and concepts. On the other hand, Algebraic problems will help students to apply algebraic manipulations, problems based on Binomial coefficients assess students' ability to interpret the same for approximation. Numeric problems require the student perform calculations or computations. Real-World Applications present realistic problem scenarios.

EXERCISES

Multiple Choice Questions

1. If the best and the worst paper never appear together, then six examination papers can be arranged in how many ways

(a) 120 (b) 480

(c) 240 (d) None of these

2. How many numbers divisible by 5 and lying between 3000 and 4000 can be formed from the digits 1, 2, 3, 4, 5, 6 (repetition is not allowed)

(a) $\frac{n+r-1}{r}$ (b) ${}^5P_2$

(c) ${}^4P_2$ (d) ${}^6P_3$

3. The number of ways in which 6 rings can be worn on the four fingers of one hand is

(a) 4^6 (b) ${}^6C_4$

(c) 6^4 (d) None of these

4. How many numbers can be formed from the digits 1, 2, 3, 4 when the repetition is not allowed

(a) ${}^4P_4$ (b) ${}^4P_3$

(c) ${}^4P_1 + {}^4P_2 + {}^4P_3$ (d) ${}^4P_1 + {}^4P_2 + {}^4P_3 + {}^4P_4$

5. There are 3 candidates for a post and one is to be selected by the votes of 7 men. The number of ways in which votes can be given is

(a) 7^3 (b) 3^7

(c) ${}^7C_3$ (d) None of these

6. 4 buses run between Bhopal and Gwalior. If a man goes from Gwalior to Bhopal by a bus and comes back to Gwalior by another bus, then the total possible ways are

(a) 12 (b) 16

(c) 4 (d) 8

7. If ${}^nP_5 = 20.\ {}^nP_3$, then $n =$

(a) 4 (b) 8

(c) 6 (d) 7

8. How many words comprising of any three letters of the word UNIVERSAL can be formed

(a) 504 (b) 405

(c) 540 (d) 450

9. If ${}^nP_4 : {}^nP_5 = 1:2$, then $n =$

(a) 4 (b) 5

(c) 6 (d) 7

10. In how many ways can mn letters be posted in n letter-boxes

(a) $(mn)^n$ (b) m^{mn}

(c) n^{mn} (d) None of these

11. In how many ways can 10 true-false questions be replied

(a) 20 (b) 100

(c) 512 (d) 1024

12. How many even numbers of 3 different digits can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 (repetition is not allowed)

(a) 224 (b) 280

(c) 324 (d) None of these

13. If ${}^{n}P_5 = 9 \times {}^{n-1}P_4$, then the value of n is

(a) 6 (b) 8

(c) 5 (d) 9

14. The value of ${}^{n}P_r$ is equal to

(a) ${}^{n-1}P_r + r\,{}^{n-1}P_{r-1}$ (b) $n.\,{}^{n-1}P_r + {}^{n-1}P_{r-1}$

(c) $n({}^{n-1}P_r + {}^{n-1}P_{r-1})$ (d) ${}^{n-1}P_{r-1} + {}^{n-1}P_r$

15. Find the total number of 9-digit numbers which have all the digits different

(a) $9 \times 9!$ (b) $9!$

(c) 10! (d) None of these

16. Four dice (six faced) are rolled. The number of possible outcomes in which at least one die shows 2 is

(a) 1296 (b) 625

(c) 671 (d) None of these

17. There are 4 parcels and 5 post-offices. In how many different ways the registration of parcel can be made

(a) 20 (b) 4^5

(c) 5^4 (d) $5^4 - 4^5$

18. In how many ways can 5 prizes be distributed among four students when every student can take one or more prizes

(a) 1024 (b) 625

(c) 120 (d) 600

19. In a train five seats are vacant, then how many ways can three passengers sit

(a) 20 (b) 30

(c) 10 (d) 60

20. The figures 4, 5, 6, 7, 8 are written in every possible order. The number of numbers greater than 56000 is

(a) 72 (b) 96

(c) 90 (d) 98

21. There are 5 roads leading to a town from a village. The number of different ways in which a villager can go to the town and return back, is

(a) 25 (b) 20

(c) 10 (d) 5

22. In how many ways can five examination papers be arranged so that physics and chemistry papers never come together

(a) 31 (b) 48

(c) 60 (d) 72

23. The number of ways in which first, second and third prizes can be given to 5 competitors is

(a) 10 (b) 60

(c) 15 (d) 125

24. The number of 3-digit odd numbers, that can be formed by using the digits 1, 2, 3, 4, 5, 6 when the repetition is allowed, is

(a) 60 (b) 108

(c) 36 (d) 30

25. How many numbers of five digits can be formed from the numbers 2, 0, 4, 3, 8 when repetition of digits is not allowed

(a) 96 (b) 120

(c) 144 (d) 14

26. If $12P_r = 1320$, then r is equal to

(a) 5 (b) 4

(c) 3 (d) 2

27. Assuming that no two consecutive digits are same, the number of n digit numbers, is

(a) $n!$ (b) $9!$

(c) 9^n (d) n^9

28. The numbers of arrangements of the letters of the word SALOON, if the two O's do not come together, is

(a) 360 (b) 720

(c) 240 (d) 120

29. The number of words which can be formed from the letters of the word MAXIMUM, if two consonants cannot occur together, is

(a) $4!$ (b) $3! \times 4!$

(c) $7!$ (d) None of these

30. How many words can be made from the letters of the word COMMITTEE

(a) $\frac{9!}{(2!)^2}$ (b) $\frac{9!}{(2!)^3}$

(c) $\frac{9!}{2!}$ (d) $9!$

31. The letters of the word MODESTY are written in all possible orders and these words are written out as in a dictionary, then the rank of the word MODESTY is

(a) 5040 (b) 720

(c) 1681 (d) 2520

32. In how many ways n books can be arranged in a row so that two specified books are not together

(a) $n!-(n-2)!$ (b) $(n-1)!(n-2)$

(c) $n!-2(n-1)$ (d) $(n-2)n!$

33. How many numbers lying between 500 and 600 can be formed with the help of the digits 1, 2, 3, 4, 5, 6 when the digits are not to be repeated

(a) 20 (b) 40

(c) 60 (d) 80

34. Numbers greater than 1000 but not greater than 4000 which can be formed with the digits 0, 1, 2, 3, 4 (repetition of digits is allowed), are

(a) 350 (b) 375

(c) 450 (d) 576

35. The number of numbers that can be formed with the help of the digits 1, 2, 3, 4, 3, 2, 1 so that odd digits always occupy odd places, is

(a) 24 (b) 18

(c) 12 (d) 30

36. In how many ways can 5 boys and 3 girls sit in a row so that no two girls are together

(a) $5! \times 3!$ (b) ${}^4P_3 \times 5!$

(c) ${}^6P_3 \times 5!$ (d) ${}^5P_3 \times 3!$

37. How many numbers less than 1000 can be made from the digits 1, 2, 3, 4, 5, 6 (repetition is not allowed)

(a) 156 (b) 160

(c) 150 (d) None of these

38. The number of diagonals in an octagon will be

(a) 28 (b) 20

(c) 10 (d) 16

39. If a polygon has 44 diagonals, then the number of its sides are

(a) 7 (b) 11

(c) 8 (d) None of these

40. How many triangles can be formed by joining four points on a circle

(a) 4 (b) 6

(c) 8 (d) 10

41. $(\sqrt{2}+1)^6-(\sqrt{2}-1)^6=$

(a) 101 (b) $70\sqrt{2}$

(c) $140\sqrt{2}$ (d) $120\sqrt{2}$

42. $x^5+10x^4a+40x^3a^2+80x^2a^3+80xa^4+32a^5=$

(a) $(x+a)^5$ (b) $(3x+a)^5$

(c) $(x+2a)^5$ (d) $(x+2a)^3$

43. The formulae

$(a+b)^m=a^m+ma^{m-1}b+\dfrac{m(m-1)}{1.2}a^{m-2}b^2+....$ holds when

(a) $b<a$ (b) $a<b$

(c) $|a|<|b|$ (d) $|b|<|a|$

44. The total number of terms in the expansion of $(x+a)^{100}+(x-a)^{100}$ after simplification will be

(a) 202 (b) 51

(c) 50 (d) None of these

45. $\dfrac{1}{\sqrt{5+4x}}$ can be expanded by binomial theorem, if

(a) $x<1$ (b) $|x|<1$

(c) $|x|<\dfrac{5}{4}$ (d) $|x|<\dfrac{4}{5}$

46. If the coefficients of r^{th} term and $(r+4)^{th}$ term are equal in the expansion of $(1+x)^{20}$, then the value of r will be

(a) 7 (b) 8

(c) 9 (d) 10

47. If x^4 occurs in the r^{th} term in the expansion of $\left(x^4+\dfrac{1}{x^3}\right)^{15}$, then $r=$

(a) 7 (b) 8

(c) 9 (d) 10

48. 16^{th} term in the expansion of $(\sqrt{x}-\sqrt{y})^{17}$ is

(a) $136xy^7$ (b) $136xy$

(c) $-136xy^{15/2}$ (d) $-136xy^2$

49. If the third term in the binomial expansion of $(1+x)^m$ is $-\frac{1}{8}x^2$, then the rational value of m is

(a) 2 (b) 1/2

(c) 3 (d) 4

50. If A and B are the coefficients of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, then

(a) $A=B$ (b) $A=2B$

(c) $2A=B$ (d) None of these

51. In the expansion of $\left(y^2+\frac{c}{y}\right)^5$, the coefficient of y will be

(a) $20c$ (b) $10c$

(c) $10c^3$ (d) $20c^2$

52. In the expansion of $\left(x-\frac{1}{x}\right)^6$, the constant term is

(a) – 20 (b) 20

(c) 30 (d) – 30

53. The middle term in the expansion of $\left(x+\frac{1}{x}\right)^{10}$ is

(a) $10C_4\frac{1}{x}$ (b) $10C_5$

(c) $10C_5x$ (d) $10C_7x^4$

Answers of Multiple-Choice Questions									
1.	b	**2.**	c	**3.**	a	**4.**	d	**5.**	b
6	a	**7.**	b	**8.**	a	**9.**	c	**10.**	c
11.	d	**12.**	a	**13.**	d	**14.**	a	**15.**	a
16.	c	**17.**	c	**18.**	a	**19.**	d	**20.**	c
21.	a	**22.**	d	**23.**	b	**24.**	b	**25.**	a
26.	c	**27.**	a	**28.**	c	**29.**	a	**30.**	b
31.	c	**32.**	b	**33.**	a	**34.**	b	**35.**	b
36.	c	**37.**	a	**38.**	b	**39.**	b	**40.**	a
41.	c	**42.**	c	**43.**	d	**44.**	b	**45.**	c
46.	c	**47.**	c	**48.**	c	**49.**	b	**50.**	b
51.	c	**52.**	a	**53.**	b				

Short and Long Answer Type Questions

1. Find the value of the following.

(i) $(x^0)!$ (ii) $({}^{101}C_1)!$

(iii) $(Log1)!$ (vi) $5!$

(v) $3 \times 4!$ (vi) $6 \times 5 \times 4!$

2. Find the value of the following.

(i) ${}^{8}P_5$ (ii) ${}^{14}P_2$

(iii) $P(901,1)$ (iv) ${}^{12}C_3$

(v) ${}^{9}C_5$ (vi) ${}^{2001}C_0$

3. Find the value of the following.

(i) $\frac{10!}{5!4!}$ (ii) $\frac{8!-3!}{6!}$

(iii) $6!+4!$

4. Find k, if $6\left(\frac{1}{12!}+\frac{1}{13!}\right)=\frac{k}{12!+11!}$

5. If $(n-1)!+n!+576=(n+1)!$ find n.

6. In how many ways can 3 different prizes be awarded to 10 students, without giving both to the same student?

7. There are 6 doors in a seminar hall, in how many ways can a person enter the hall through a door and leave it by a different door?

8. From Bhopal to Mumbai, there are three routs: road, rail and air. From Mumbai to Surat there are four routs: road, rail, air and see. How many routs are there from Bhopal to Surat?

9. How many different numbers of 3-digits can be formed with the digits 5,6,7,8,9 (No digit being repeated in the same number)?

10. How many words (with or without meaning) can be formed using the letters of the word "OXYGEN"?

11. Prove that, ${}^{12}P_4 = {}^{11}P_4 + 4\left({}^{11}P_3\right)$

12. Find r, if $P(8,r)=8P(9,r-1)$

13. How many different words can be formed out of the letters of the word "COMPLIANT" so that vowels never occur together?

14. How many numbers greater than 30000 can be formed by using the digits 0, 1, 2, 3, 4, 5, no digit being repeated in any number?

15. In how many different ways can the letter "TRIGONOMETRY" be arranged such that consonants occur together?

16. In how many ways can 6 women and 6 men be seated at around table so that No two women are together?

17. If ${}^{2n}C_5 : {}^{n}C_5 = 286:3$ then find n.

18. Verify $r\left({}^{n}C_r\right) = n\left({}^{n-1}C_{r-1}\right)$.

19. A committee of 5 is to be found from 5 boys and 4 girls. In how many ways can this be done if the committee contains (i) 2 boys (ii) at least 2 girls?

20. A bag contains 6 black and 5 red bolls,6 balls are drawn. Determine the number of ways in which 3 black and 3 red balls can be drawn?

21. In how many ways can 11 players be select from a group of 15 players?

22. Find the 6^{th} term in the expansion of $\left(x+\frac{1}{y}\right)^{11}$.

23. Find the 3^{rd} term from the end in the expansion of $\left(\frac{1}{x}-3x\right)^{6}$.

24. Find the term independent of x in the expansion of $\left(3x^2+\frac{1}{2x^2}\right)^{8}$

25. Find the middle term in the expansion of.

(i) $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^{8}$ (ii) $\left(x+\frac{1}{x}\right)^{2n}$

26. Find the middle terms in the expansion of.

(i) $\left(x^2+\frac{2}{x^3}\right)^{7}$ (ii) $\left(\frac{4}{x^3}-\frac{x^3}{2}\right)^{5}$

27. Find the coefficient of x in the expansion of $(2x+5)^5$.

28. Find the coefficient of x^{-3} in the expansion of $\left(2x^2-\frac{3}{x}\right)^{9}$.

29. Expand $\left(1-x+x^2\right)^4$ in powers of x using Binomial Theorem.

30. Simplify $(x+y)^5+(x-y)^5$ and hence evaluate $\left(\sqrt{3}+1\right)^5+\left(\sqrt{3}-1\right)^5$.

31. Using Binomial theorem, prove that $4^n - 3n - 1$ is always divisible by 9, where $n \in N$.

32. Find the reminder 3^{99} is divided by 5.

33. Using Binomial theorem, evaluate each of the following.

(i) $(1.05)^4$ (ii) $(99.01)^3$

Answers of Short and Long Answer Type Questions						
1.	(i) 1	(ii) 1	(iii) 1	(iv) 120	(v) 72	(vi) 720
2.	(i) 6720	(ii) 182	(iii) 901	(iv) 220	(v) 126	(vi) 1
3.	(i) 1260	(ii) 126	(iii) 744			
4.	$k = 7$					
5.	$n = 5$					
6.	720					
7.	30					
8.	12					
9.	60					
10.	720					
12.	1					
13.	30240					
14.	360					
15.	604800					
16.	6!5!					
17.	14					
19.	(i) 40	(ii) 105				
20.	200					
21.	1365					
22.	$462\dfrac{x^6}{y^6}$					
23.	$1215x^4$					
24.	$\dfrac{2835}{8}$					
25.	(i) $-\dfrac{7}{9}x$	(ii) $\dfrac{(2n)!}{n!n!}$				

Answers of Short and Long Answer Type Questions		
26.	(i) $\frac{280}{x}, \frac{560}{x^6}$	(ii) $-\frac{1120}{x^3}, 140x^3$
27.	6250	
28.	−314928	
29.	$1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$	
30.	$88\sqrt{3}$	
32.	7	
33.	(i) 1.21550625	(ii) 97059305.9701

KNOW MORE

- Why Study Permutations and Combinations?
- How Binomial theorem and its coefficients are used in our daily lives.
- Use of mathematical reasoning by justifying and generalizing patterns and relationships.
- Development of mathematics in historical context with contemporary non-mathematical events.
- How problems based on mathematics can be used unfamiliar settings.
- Why mathematical thinking is valuable in daily life
- Shift to Online Teaching.
- The simplest way to learn Permutations and combinations and its algebra.
- Why was Theory of Binomial theorem and its coefficients invented?
- Learning of Permutation and combinations intuitively.
- Making of calculations less complicated.
- Online Education Tools for Teachers.
- Teaching Critical Thinking
- STEM Education.

Mini Project

i. Prepare a micro project on patterns in Pascal's Triangle, such as horizontal sums and patterns that form Fibonacci sequence.

ii. Prepare a mini project on applications of Binomial theorem to Machine learning & IOT (Internet of things).

Inquisitiveness and Curiosity Topics

i. In a Society of 8 members we have to select a committee of 3 members, as the owner of the society Vikas is already a member of the committee in how many ways the committee can be found?

ii. If I make a cup of tea with a combination of sugar, water, milk and tea, whether order matters?

iii. How many different ways can you arrange 7 planets?

iv. One day, I wanted to travel from Pune to Jabalpur by train. There is no direct train from Pune to Jabalpur, but there are trains from Pune to Bhopal and from Bhopal to Jabalpur. There were three trains from Pune to Bhopal and four trains from Bhopal to Jabalpur. Now, in how many ways can I travel from Pune to Jabalpur

v. In a Sport event suppose five teams are contending. First place gets gold and second place gets silver medals. How many distinct ways can medals be given to these teams?

vi. In a Sport event suppose seven teams are competing. First place gets 'A' type medal and second place gets 'B' type medal. How many groups of medal winners are possible? Order of teams doesn't matter.

vii. From a committee of 11 people. In how many can we choose a chair person, a vice-chair person, a secretary and a treasurer, assuming that one-person cannot hold more than one position?

viii. Binomial probability distributions help us to comprehend the possibility of rare events and to set possible predictable ranges, comment on it.

ix. How the Binomial theorem can be used in performance of a machine learning model?

Apart from the above questions, Permutation and Combination and Binomial theorem can be utilized for the real-world problems from the domain of Economy, Higher mathematics, forecast services, Ranking, Internet protocol (IP), Architecture, Finance, Population estimation and many more.

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APPENDICES

APPENDIX: Assessments Aligned to Bloom's Level

Bloom's Taxonomy – It has been coupled into following two categories for development of Questions for this Quadrant as given below:

Category I Questions	Category II Questions *- Higher Order Thinking Skills*
Bloom's Level 1: Remember Bloom's Level 2: Understand Bloom's Level 3: Apply	Bloom's Level 4: Analyze Bloom's Level 5: Evaluate Bloom's Level 6: Create

SAMPLE SPECIFICATION TABLE

Course Outcomes Number	Unit Number	Unit Titles	Marks Distribution			Total Marks
			R	U	A	
CO-1	I	Trigonometry	2	4	6	12
CO-2	II	Functions and Limit	2	4	4	10
CO-3	III	Differential Calculus	2	8	10	20
CO-4 CO-5 CO-6	IV	Complex numbers and Partial Fraction	2	4	8	14
CO-7	V	Permutation and Combination, Binomial Theorem	2	6	6	14
	Total		**10**	**26**	**34**	**70**

REFERENCES FOR FURTHER LEARNING

Lists of some of the books are given below which may be used for further learning of the subject (both theory and practical) by the interested students:

1. B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, New Delhi, 40th Edition, 2007.
2. G. B. Thomas, R. L. Finney, Calculus and Analytic Geometry, Addison Wesley, 9th Edition, 1995.
3. Reena Garg, Engineering Mathematics, Khanna Publishing House, New Delhi (Revised Ed. 2018). 4. V. Sundaram, R. Balasubramanian, K.A. Lakshminarayanan, Engineering Mathematics, 6/e., Vikas Publishing House.
5. Reena Garg & Chandrika Prasad, Advanced Engineering Mathematics, Khanna Publishing House, New Delhi

CO AND PO ATTAINMENT TABLE

Course outcomes (COs) for this course can be mapped with the programme outcomes (POs) after the completion of the course and a correlation can be made for the attainment of POs to analyze the gap. After proper analysis of the gap in the attainment of POs necessary measures can be taken to overcome the gaps.

Table for CO and PO attainment

Course Outcomes	Attainment of Programme Outcomes *(1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation)*						
	PO-1	PO-2	PO-3	PO-4	PO-5	PO-6	PO-7
CO-1							
CO-2							
CO-3							
CO-4							
CO-5							
CO-6							
CO-7							

The data filled in the above table can be used for gap analysis.

INDEX

Algebra of derivative of function, 66
Algebraic operation of complex numbers, 94
Allied angles, 6
Angles, 2
Approximations by Binomial Theorem, 135
Argument (amplitude) of complex number, 101
Binomial expression, 133
Binomial theorem for any index, 134
Binomial theorem for positive integral index, 133
Centesimal system, 3
Chain rule, 70
Closed interval, 44
Co-domain of function, 40
Combinations, 131
Complex Number - definition, 93
Conjugate of complex number, 97
Constant function, 41
Conversions of one form into another, 104
Degree, 3
De-Moivre's Theorem, 105
Derivative of $\log_e x$ by definition, 65
Derivative of composite function, 70
Derivative of cos x by definition, 64
Derivative of division, 66
Derivative of e^x by definition, 64
Derivative of function at a point, 62
Derivative of function, 63
Derivative of product with constant, 66
Derivative of product, 66
Derivative of sin x by definition, 63
Derivative of sum/difference, 66
Derivative of tan x by definition, 64
Differentiation of algebraic functions, 65
Differentiation of exponential functions, 75
Differentiation of inverse-trigonometric functions, 72
Differentiation of logarithmic functions, 75
Differentiation of standard functions by definition, 63
Differentiation of trigonometric functions, 72
Domain of function, 40
Equality of complex numbers, 96
Even function, 44
Existence of limit, 45
Exponential function, 43
Factorial notation, 127
First principal of derivative, 63
Function, 38
Fundamental principal of counting, 124
General term, 134
Geometrical /Cartesian representation, 103
Grade, 3
Graphs of cos x, 21
Graphs of e^x, 22
Graphs of sin x, 21
Graphs of tan x, 21
Graphs of the function, 20
Greatest integer function, 42
Identity function, 41
Imaginary part, 94

Improper fractions, 112
Integral powers of i(iota), 93
Intervals, 44
iota, 93
Left and right limit, 45
Limit of a function, 44
Modulus function, 42
Modulus of complex number, 99
Multiple angles, 17
Odd function, 44
Open interval, 44
Product into Sum or difference, 23
Proper fractions, 112
Properties of algebraic operations, 94
Properties of conjugate, 97
Properties of logarithm, 43
Properties of modulus, 99
Partial Fractions, 107
Permutations when all objects are distinct, 126
Permutations, 126
Principle of addition, 126
Principle of multiplication, 124
Principle value of argument of complex, 101
Radian, 3
Range of function, 41
Real part, 94
Reciprocal function, 43
Semi-open or Semi-closed interval, 44
Sexagesimal system, 3
Signum function, 43
Some important Binomial expansions, 137
Sub-multiple angles, 17
Sum and difference formulae, 10
Sum or difference into Product Formulae, 13
System of measurement of angles, 3
t-ratios *of* $(180° - \theta)$, 7
t-ratios *of* $(90° - \theta)$, 7
t-ratios *of* $(A/2)$, 17
t-ratios *of* $(-\theta)$, 7
t-ratios *of* $2A$, 17
t-ratios *of* $3A$, 17
Trigonometrical (Polar) representation, 104
Trigonometrical ratios of allied angles, 6
Unimodular complex number, 99
Value of arguments of complex number, 102