UNIT-I

NETWORK THEOREMS

Introduction to network theorems

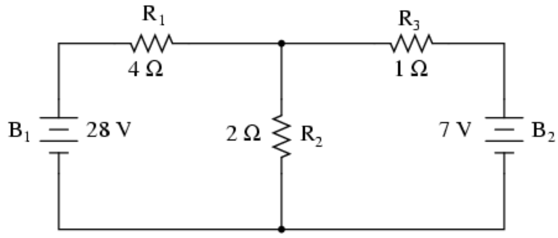
Anyone who has studied geometry should be familiar with the concept of a *theorem*: a relatively simple rule used to solve a problem, derived from a more intensive analysis using fundamental rules of mathematics. At least hypothetically, any problem in math can be solved just by using the simple rules of arithmetic (in fact, this is how modern digital computers carry out the most complex mathematical calculations: by repeating many cycles of additions and subtractions!), but human beings aren't as consistent or as fast as a digital computer. We need “shortcut” methods in order to avoid procedural errors.

In electric network analysis, the fundamental rules are Ohm's Law and Kirchhoff's Laws. While these humble laws may be applied to analyze just about any circuit configuration (even if we have to resort to complex algebra to handle multiple unknowns), there are some “shortcut” methods of analysis to make the math easier for the average human. As with any theorem of geometry or algebra, these network theorems are derived from fundamental rules.

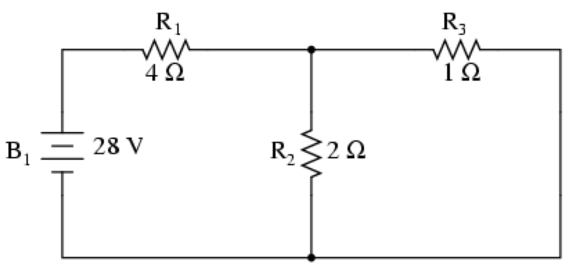
Superposition Theorem

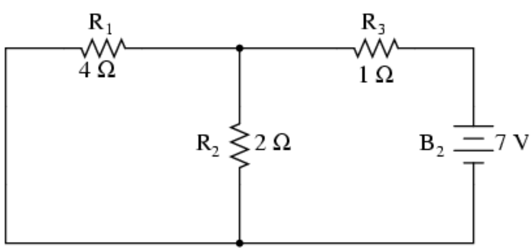
Superposition theorem is one of those strokes of genius that takes a complex subject and simplifies it in a way that makes perfect sense.The strategy used in the Superposition Theorem is to eliminate all but one source of power within a network at a time, using series/parallel analysis to determine voltage drops (and/or currents) within the modified network for each power source separately. Then, once voltage drops and/or currents have been determined for each power source working separately, the values are all “superimposed” on top of each other (added algebraically) to find the actual voltage drops/currents with all sources active.

Consider a circuit shown below and apply Superposition Theorem to it:



Since we have two sources of power in this circuit, we will have to calculate two sets of values for voltage drops and/or currents, one for the circuit with only the 28 volt battery in effect and one for the circuit with only the 7 volt battery in effect:

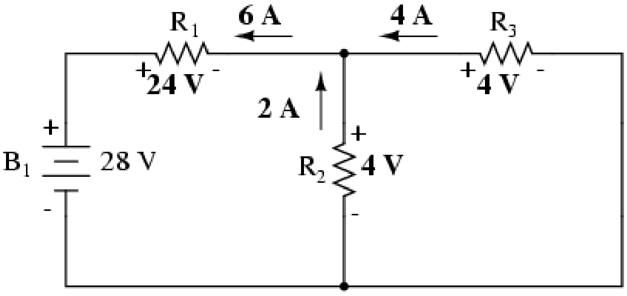




When re-drawing the circuit for series/parallel analysis with one source, all other voltage sources are replaced by wires (shorts), and all current sources with open circuits (breaks). Since we only have voltage sources (batteries) in our example circuit, we will replace every inactive source during analysis with a wire.

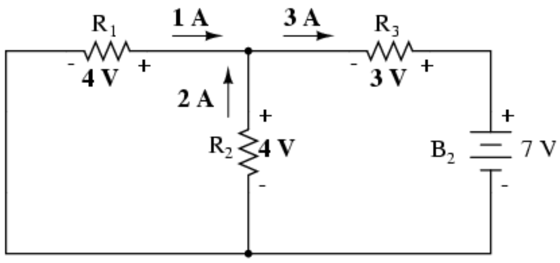
Analyzing the circuit with only the 28 volt battery, we obtain the following values for voltage and current:

	R_1	R_2	R_3	$R_2//R_3$	$R_1 + R_2//R_3$ Total	
E	24	4	4	4	28	Volts
I	6	2	4	6	6	Amps
R	4	2	1	0.667	4.667	Ohms



Analyzing the circuit with only the 7 volt battery, we obtain another set of values for voltage and current:

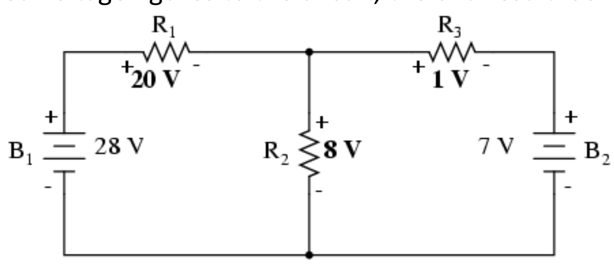
	R_1	R_2	R_3	$R_1//R_2$	$R_3 + R_1//R_2$ Total	
E	4	4	3	4	7	Volts
I	1	2	3	3	3	Amps
R	4	2	1	1.333	2.333	Ohms



When superimposing these values of voltage and current, we have to be very careful to consider polarity (voltage drop) and direction (electron flow), as the values have to be added *algebraically*.

With 28 V battery	With 7 V battery	With both batteries
24 V (+ −) E_{R1}	4 V (− +) E_{R1}	20 V (+ −) E_{R1} *24 V - 4 V = 20 V*
E_{R2} 4 V (+ −)	E_{R2} 4 V (+ −)	E_{R2} **8 V** (+ −) *4 V + 4 V = 8 V*
4 V (+ −) E_{R3}	3 V (− +) E_{R3}	1 V (+ −) E_{R3} *4 V - 3 V = 1 V*

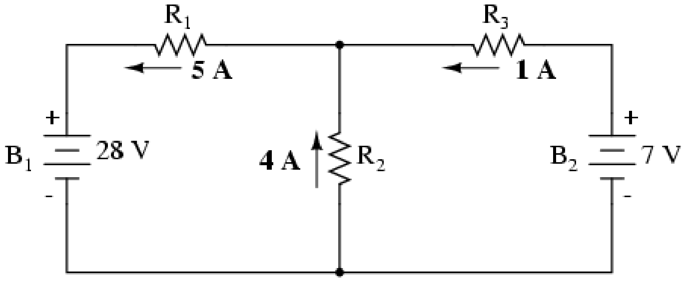
Applying these superimposed voltage figures to the circuit, the end result looks something like this:



Currents add up algebraically as well, and can either be superimposed as done with the resistor voltage drops, or simply calculated from the final voltage drops and respective resistances (I=E/R). Either way, the answers will be the same. Here I will show the superposition method applied to current:

With 28 V battery	With 7 V battery	With both batteries
← 6 A, I_{R1}	→ 1 A, I_{R1}	← **5 A**, I_{R1} $6 A - 1 A = 5 A$
I_{R2} ↑ 2 A	I_{R2} ↑ 2 A	I_{R2} ↑ **4 A** $2 A + 2 A = 4 A$
← 4 A, I_{R3}	→ 3 A, I_{R3}	← **1 A**, I_{R3} $4 A - 3 A = 1 A$

Once again applying these superimposed figures to our circuit:



Quite simple and elegant, don't you think? It must be noted, though, that the Superposition Theorem works only for circuits that are reducible to series/parallel combinations for each of the power sources at a time (thus, this theorem is useless for analyzing an unbalanced bridge circuit), and it only works where the underlying equations are linear (no mathematical powers or roots). The requisite of linearity means that Superposition Theorem is only applicable for determining voltage and current, *not power!!!* Power dissipations, being nonlinear functions, do not algebraically add to an accurate total when only one source is considered at a time. The need for linearity also means this Theorem cannot be applied in circuits where the resistance of a component changes with voltage or current. Hence, networks containing components like lamps (incandescent or gas-discharge) or varistors could not be analyzed.

Another prerequisite for Superposition Theorem is that all components must be “bilateral,” meaning that they behave the same with electrons flowing either direction through them. Resistors have no polarity-specific behavior, and so the circuits we've been studying so far all meet this criterion.

The Superposition Theorem finds use in the study of alternating current (AC) circuits, and semiconductor (amplifier) circuits, where sometimes AC is often mixed (superimposed) with DC. Because AC voltage and current equations (Ohm's Law) are linear just like DC, we can use Superposition to analyze the circuit with just the DC power source, then just the AC power source, combining the results to tell what will happen with both AC and DC sources in effect. For now, though, Superposition will suffice as a break from having to do simultaneous equations to analyze a circuit.

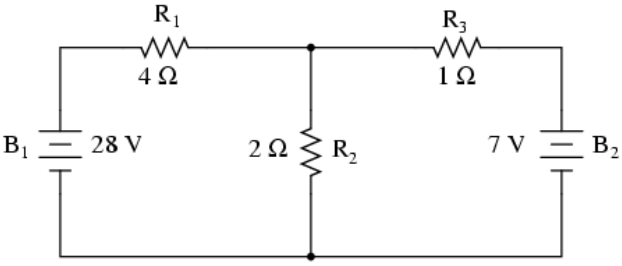
REVIEW:

- The Superposition Theorem states that a circuit can be analyzed with only one source of power at a time, the corresponding component voltages and currents algebraically added to find out what they'll do with all power sources in effect.
- To negate all but one power source for analysis, replace any source of voltage (batteries) with a wire; replace any current source with an open (break).

Thevenin's Theorem

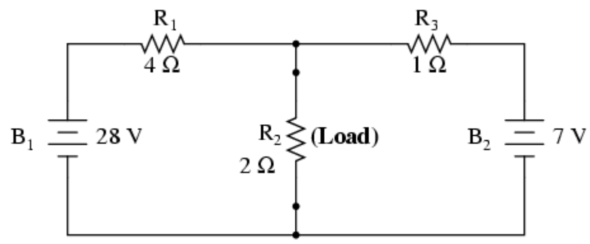
Thevenin's Theorem states that it is possible to simplify any linear circuit, no matter how complex, to an equivalent circuit with just a single voltage source and series resistance connected to a load. The qualification of “linear” is identical to that found in the Superposition Theorem, where all the underlying equations must be linear (no exponents or roots). If we're dealing with passive components (such as resistors, and later, inductors and capacitors), this is true. However, there are some components (especially certain gas-discharge and semiconductor components) which are nonlinear: that is, their opposition to current *changes* with voltage and/or current. As such, we would call circuits containing these types of components, *nonlinear circuits*.

Thevenin's Theorem is especially useful in analyzing power systems and other circuits where one particular resistor in the circuit (called the “load” resistor) is subject to change, and re-calculation of the circuit is necessary with each trial value of load resistance, to determine voltage across it and current through it. Let's take another look at our example circuit:

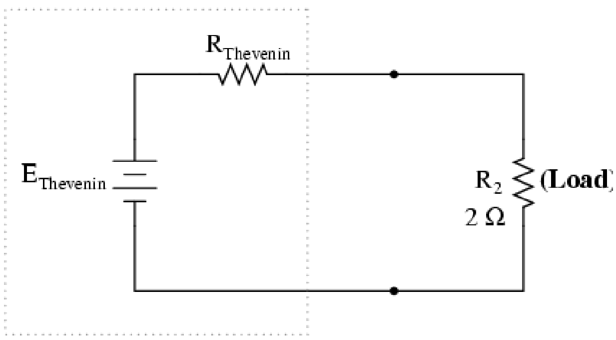


Let's suppose that we decide to designate R_2 as the “load” resistor in this circuit. We already have four methods of analysis at our disposal (Branch Current, Mesh Current, Millman's Theorem, and Superposition Theorem) to use in determining voltage across R_2 and current through R_2, but each of these methods are time-consuming. Imagine repeating any of these methods over and over again to find what would happen if the load resistance changed (changing load resistance is *very* common in power systems, as multiple loads get switched on and off as needed. the total resistance of their parallel connections changing depending on how many are connected at a time). This could potentially involve a *lot* of work!

Thevenin's Theorem makes this easy by temporarily removing the load resistance from the original circuit and reducing what's left to an equivalent circuit composed of a single voltage source and series resistance. The load resistance can then be re-connected to this “Thevenin equivalent circuit” and calculations carried out as if the whole network were nothing but a simple series circuit: after Thevenin conversion



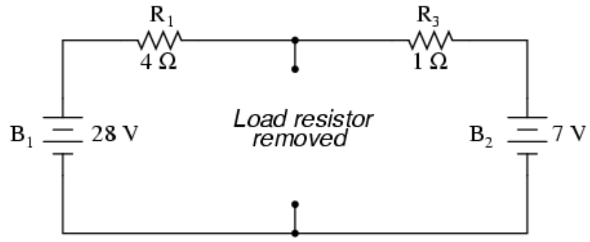
Thevenin Equivalent Circuit



The “Thevenin Equivalent Circuit” is the electrical equivalent of B_1, R_1, R_3, and B_2 as seen from the two points where our load resistor (R_2) connects.

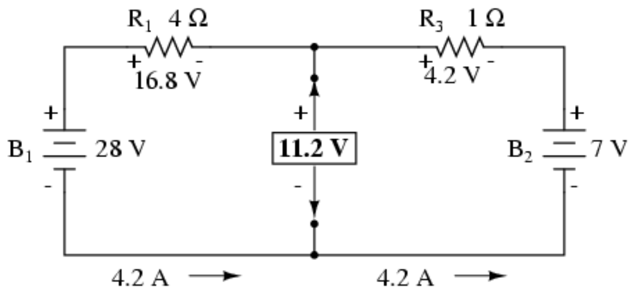
The Thevenin equivalent circuit, if correctly derived, will behave exactly the same as the original circuit formed by B_1, R_1, R_3, and B_2. In other words, the load resistor (R_2) voltage and current should be exactly the same for the same value of load resistance in the two circuits. The load resistor R_2 cannot “tell the difference” between the original network of B_1, R_1, R_3, and B_2, and the Thevenin equivalent circuit of $E_{Thevenin}$, and $R_{Thevenin}$, provided that the values for $E_{Thevenin}$ and $R_{Thevenin}$ have been calculated correctly.

The advantage in performing the “Thevenin conversion” to the simpler circuit, of course, is that it makes load voltage and load current so much easier to solve than in the original network. Calculating the equivalent Thevenin source voltage and series resistance is actually quite easy. First, the chosen load resistor is removed from the original circuit, replaced with a break (open circuit):



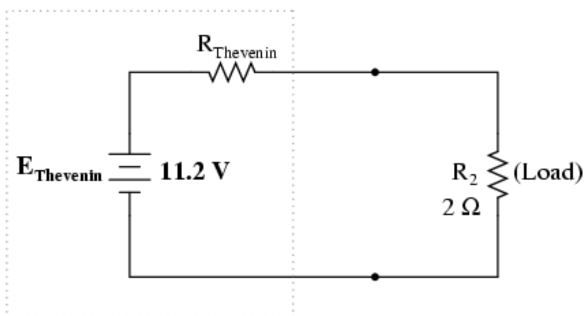
Next, the voltage between the two points where the load resistor used to be attached is determined. Use whatever analysis methods are at your disposal to do this. In this case, the original circuit with the load resistor removed is nothing more than a simple series circuit with opposing batteries, and so we can determine the voltage across the open load terminals by applying the rules of series circuits, Ohm's Law, and Kirchhoff's Voltage Law:

	R_1	R_3	Total	
E	16.8	4.2	21	Volts
I	4.2	4.2	4.2	Amps
R	4	1	5	Ohms

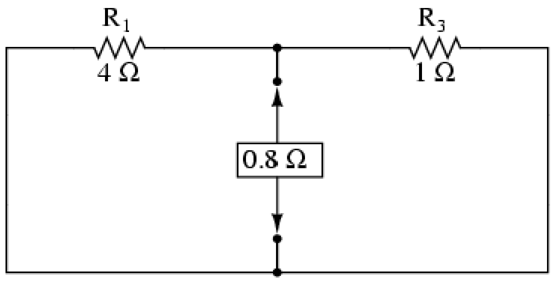


The voltage between the two load connection points can be figured from the one of the battery's voltage and one of the resistor's voltage drops, and comes out to 11.2 volts. This is our “Thevenin voltage” ($E_{Thevenin}$) in the equivalent circuit:

Thevenin Equivalent Circuit

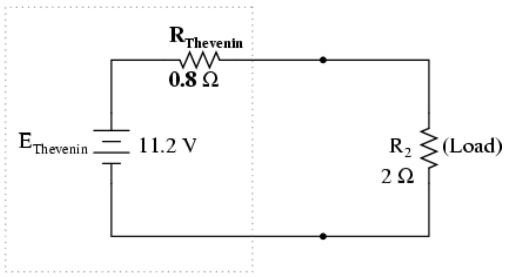


To find the Thevenin series resistance for our equivalent circuit, we need to take the original circuit (with the load resistor still removed), remove the power sources (in the same style as we did with the Superposition Theorem: voltage sources replaced with wires and current sources replaced with breaks), and figure the resistance from one load terminal to the other:



With the removal of the two batteries, the total resistance measured at this location is equal to R_1 and R_3 in parallel: 0.8 Ω. This is our "Thevenin resistance" ($R_{Thevenin}$) for the equivalent circuit:

Thevenin Equivalent Circuit



With the load resistor (2 Ω) attached between the connection points, we can determine voltage across it and current through it as though the whole network were nothing more than a simple series circuit:

	$R_{Thevenin}$	R_{Load}	Total	
E	3.2	8	11.2	Volts
I	4	4	4	Amps
R	0.8	2	2.8	Ohms

Notice that the voltage and current figures for R_2 (8 volts, 4 amps) are identical to those found using other methods of analysis. Also notice that the voltage and current figures for the Thevenin series resistance and the Thevenin source (*total*) do not apply to any component in the original, complex circuit. Thevenin's Theorem is only useful for determining what happens to a *single* resistor in a network: the load.

The advantage, of course, is that you can quickly determine what would happen to that single resistor if it were of a value other than 2 Ω without having to go through a lot of analysis again. Just plug in that other value for the load resistor into the Thevenin equivalent circuit and a little bit of series circuit calculation will give you the result.

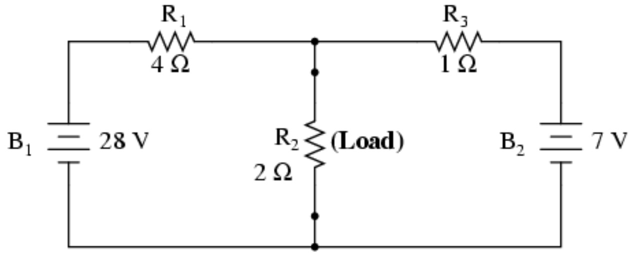
REVIEW:

- Thevenin's Theorem is a way to reduce a network to an equivalent circuit composed of a single voltage source, series resistance, and series load.
- Steps to follow for Thevenin's Theorem:
- (1) Find the Thevenin source voltage by removing the load resistor from the original circuit and calculating voltage across the open connection points where the load resistor used to be.
- (2) Find the Thevenin resistance by removing all power sources in the original circuit (voltage sources shorted and current sources open) and calculating total resistance between the open connection points.
- (3) Draw the Thevenin equivalent circuit, with the Thevenin voltage source in series with the Thevenin resistance. The load resistor re-attaches between the two open points of the equivalent circuit.
- (4) Analyze voltage and current for the load resistor following the rules for series circuits

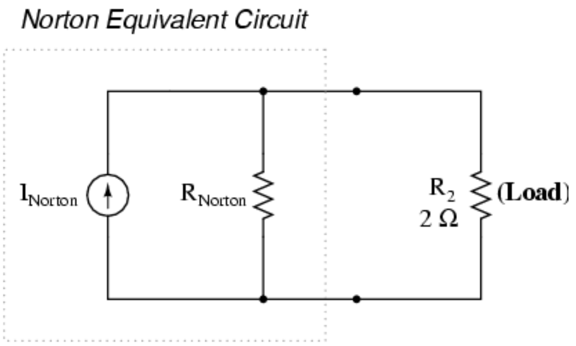
Norton's Theorem

Norton's Theorem states that it is possible to simplify any linear circuit, no matter how complex, to an equivalent circuit with just a single current source and parallel resistance connected to a load. Just as with Thevenin's Theorem, the qualification of "linear" is identical to that found in the Superposition Theorem: all underlying equations must be linear (no exponents or roots).

Contrasting our original example circuit against the Norton equivalent: it looks something like this:



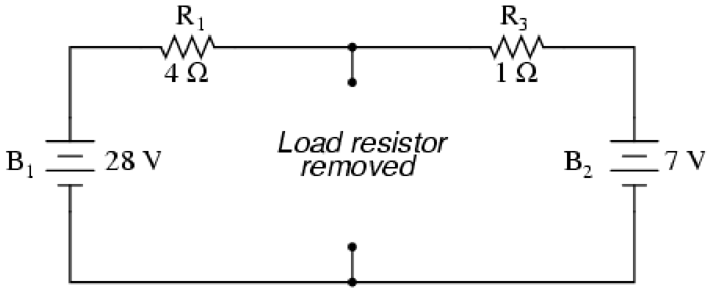
after Norton conversion



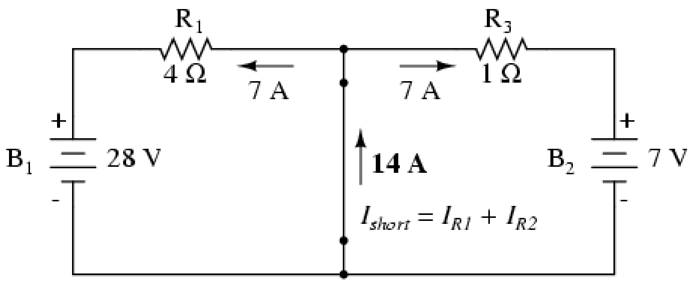
Remember that a *current source* is a component whose job is to provide a constant amount of current, outputting as much or as little voltage necessary to maintain that constant current.

As with Thevenin's Theorem, everything in the original circuit except the load resistance has been reduced to an equivalent circuit that is simpler to analyze. Also similar to Thevenin's Theorem are the steps used in Norton's Theorem to calculate the Norton source current (I_{Norton}) and Norton resistance (R_{Norton}).

As before, the first step is to identify the load resistance and remove it from the original circuit:

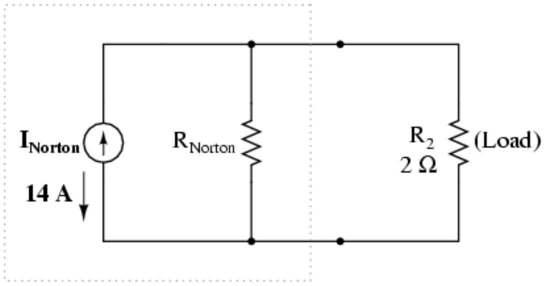


Then, to find the Norton current (for the current source in the Norton equivalent circuit), place a direct wire (short) connection between the load points and determine the resultant current. Note that this step is exactly opposite the respective step in Thevenin's Theorem, where we replaced the load resistor with a break (open circuit):

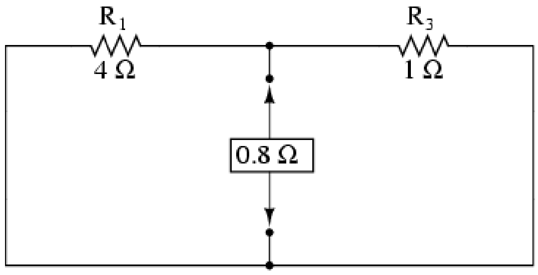


With zero voltage dropped between the load resistor connection points, the current through R_1 is strictly a function of B_1's voltage and R_1's resistance: 7 amps (I=E/R). Likewise, the current through R_3 is now strictly a function of B_2's voltage and R_3's resistance: 7 amps (I=E/R). The total current through the short between the load connection points is the sum of these two currents: 7 amps + 7 amps = 14 amps. This figure of 14 amps becomes the Norton source current (I_{Norton}) in our equivalent circuit:

Norton Equivalent Circuit

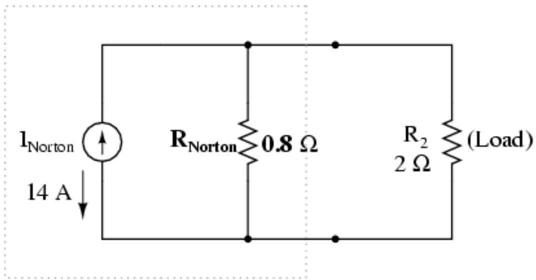


To calculate the Norton resistance (R_{Norton}), we do the exact same thing as we did for calculating Thevenin resistance ($R_{Thevenin}$): take the original circuit (with the load resistor still removed), remove the power sources (in the same style as we did with the Superposition Theorem: voltage sources replaced with wires and current sources replaced with breaks), and figure total resistance from one load connection point to the other:



Now our Norton equivalent circuit looks like this:

Norton Equivalent Circuit



If we re-connect our original load resistance of 2 Ω, we can analyze the Norton circuit as a simple parallel arrangement:

	R_{Norton}	R_{Load}	Total	
E	8	8	8	Volts
I	10	4	14	Amps
R	0.8	2	571.43m	Ohms

As with the Thevenin equivalent circuit, the only useful information from this analysis is the voltage and current values for R_2; the rest of the information is irrelevant to the original circuit. However, the same advantages seen with Thevenin's Theorem apply to Norton's as well: if we wish to analyze load resistor voltage and current over several different values of load resistance, we can use the Norton equivalent circuit again and again, applying nothing more complex than simple parallel circuit analysis to determine what's happening with each trial load.

REVIEW:

- Norton's Theorem is a way to reduce a network to an equivalent circuit composed of a single current source, parallel resistance, and parallel load.
- Steps to follow for Norton's Theorem:

- (1) Find the Norton source current by removing the load resistor from the original circuit and calculating current through a short (wire) jumping across the open connection points where the load resistor used to be.
- (2) Find the Norton resistance by removing all power sources in the original circuit (voltage sources shorted and current sources open) and calculating total resistance between the open connection points.
- (3) Draw the Norton equivalent circuit, with the Norton current source in parallel with the Norton resistance. The load resistor re-attaches between the two open points of the equivalent circuit.
- (4) Analyze voltage and current for the load resistor following the rules for parallel circuits.

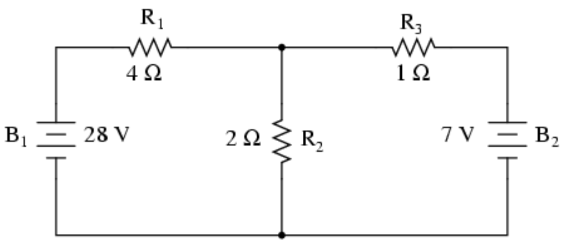
Thevenin-Norton equivalencies

Since Thevenin's and Norton's Theorems are two equally valid methods of reducing a complex network down to something simpler to analyze, there must be some way to convert a Thevenin equivalent circuit to a Norton equivalent circuit, and vice versa (just what you were dying to know, right?). Well, the procedure is very simple.

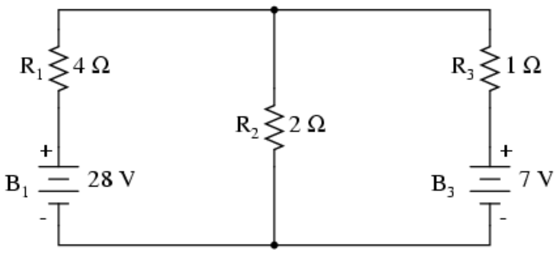
You may have noticed that the procedure for calculating Thevenin resistance is identical to the procedure for calculating Norton resistance: remove all power sources and determine resistance between the open load connection points. As such, Thevenin and Norton resistances for the same original network must be equal. Using the example circuits from the last two sections, we can see that the two resistances are indeed equal:

Millman's Theorem

In Millman's Theorem, the circuit is re-drawn as a parallel network of branches, each branch containing a resistor or series battery/resistor combination. Millman's Theorem is applicable only to those circuits which can be re-drawn accordingly. Here again is our example circuit used for the last two analysis methods:



And here is that same circuit, re-drawn for the sake of applying Millman's Theorem:



By considering the supply voltage within each branch and the resistance within each branch, Millman's Theorem will tell us the voltage across all branches. Please note that I've labeled the battery in the rightmost branch as "B_3" to clearly denote it as being in the third branch, even though there is no "B_2" in the circuit!

Millman's Theorem is nothing more than a long equation, applied to any circuit drawn as a set of parallel-connected branches, each branch with its own voltage source and series resistance:

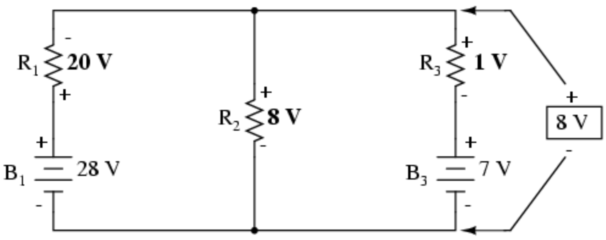
Millman's Theorem Equation

$$\frac{\dfrac{E_{B1}}{R_1} + \dfrac{E_{B2}}{R_2} + \dfrac{E_{B3}}{R_3}}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}} = \text{Voltage across all branches}$$

Substituting actual voltage and resistance figures from our example circuit for the variable terms of this equation, we get the following expression:

$$\frac{\dfrac{28\ \text{V}}{4\ \Omega}+\dfrac{0\ \text{V}}{2\ \Omega}+\dfrac{7\ \text{V}}{1\ \Omega}}{\dfrac{1}{4\ \Omega}+\dfrac{1}{2\ \Omega}+\dfrac{1}{1\ \Omega}}=8\ \text{V}$$

The final answer of 8 volts is the voltage seen across all parallel branches, like this:



The polarity of all voltages in Millman's Theorem are referenced to the same point. In the example circuit above, I used the bottom wire of the parallel circuit as my reference point, and so the voltages within each branch (28 for the R_1 branch, 0 for the R_2 branch, and 7 for the R_3 branch) were inserted into the equation as positive numbers. Likewise, when the answer came out to 8 volts (positive), this meant that the top wire of the circuit was positive with respect to the bottom wire (the original point of reference). If both batteries had been connected backwards (negative ends up and positive ends down), the voltage for branch 1 would have been entered into the equation as a -28 volts, the voltage for branch 3 as -7 volts, and the resulting answer of -8 volts would have told us that the top wire was negative with respect to the bottom wire (our initial point of reference).

To solve for resistor voltage drops, the Millman voltage (across the parallel network) must be compared against the voltage source within each branch, using the principle of voltages adding in series to determine the magnitude and polarity of voltage across each resistor:

$$E_{R1} = 8\ \text{V} - 28\ \text{V} = -20\ \text{V} \quad \text{(negative on top)}$$

$$E_{R2} = 8\ \text{V} - 0\ \text{V} = 8\ \text{V} \quad \text{(positive on top)}$$

$$E_{R3} = 8\ \text{V} - 7\ \text{V} = 1\ \text{V} \quad \text{(positive on top)}$$

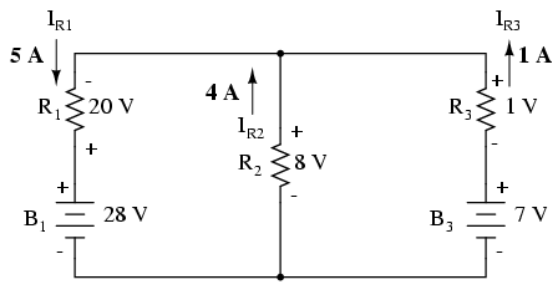
To solve for branch currents, each resistor voltage drop can be divided by its respective resistance (I=E/R):

$$I_{R1} = \frac{20\ \text{V}}{4\ \Omega} = 5\ \text{A}$$

$$I_{R2} = \frac{8\ \text{V}}{2\ \Omega} = 4\ \text{A}$$

$$I_{R3} = \frac{1\ \text{V}}{1\ \Omega} = 1\ \text{A}$$

The direction of current through each resistor is determined by the polarity across each resistor, *not* by the polarity across each battery, as current can be forced backwards through a battery, as is the case with B_3 in the example circuit. This is important to keep in mind, since Millman's Theorem doesn't provide as direct an indication of "wrong" current direction as does the Branch Current or Mesh Current methods. You must pay close attention to the polarities of resistor voltage drops as given by Kirchhoff's Voltage Law, determining direction of currents from that.



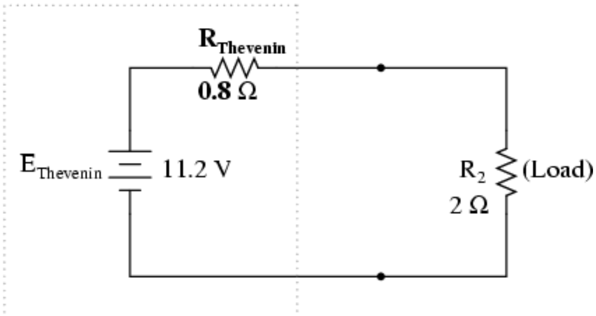
Millman's Theorem is very convenient for determining the voltage across a set of parallel branches, where there are enough voltage sources present to preclude solution via regular series-parallel reduction method. It also is easy in the sense that it doesn't require the use of simultaneous equations. However, it is limited in that it only applied to circuits which can be re-drawn to fit this form. It cannot be used, for example, to solve an unbalanced bridge circuit. And, even in cases where Millman's Theorem can be applied, the solution of individual resistor voltage drops can be a bit daunting to some, the Millman's Theorem equation only providing a single figure for branch voltage.

As you will see, each network analysis method has its own advantages and disadvantages. Each method is a tool, and there is no tool that is perfect for all jobs. The skilled technician, however, carries these methods in his or her mind like a mechanic carries a set of tools in his or her tool box. The more tools you have equipped yourself with, the better prepared you will be for any eventuality.

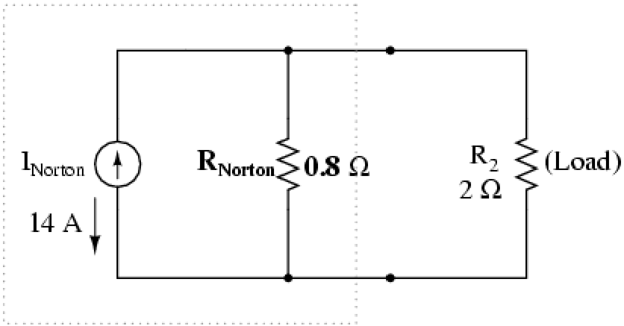
REVIEW:

- Millman's Theorem treats circuits as a parallel set of series-component branches.
- All voltages entered and solved for in Millman's Theorem are polarity-referenced at the same point in the circuit (typically the bottom wire of the parallel network).

Thevenin Equivalent Circuit



Norton Equivalent Circuit



$$R_{Thevenin} = R_{Norton}$$

Considering the fact that both Thevenin and Norton equivalent circuits are intended to behave the same as the original network in suppling voltage and current to the load resistor (as seen from the perspective of the load connection points), these two equivalent circuits, having been derived from the same original network should behave identically.

This means that both Thevenin and Norton equivalent circuits should produce the same voltage across the load terminals with no load resistor attached. With the Thevenin equivalent, the open-circuited voltage would be equal to the Thevenin source voltage (no circuit current present to drop voltage across the series resistor), which is 11.2 volts in this case. With the Norton equivalent circuit, all 14 amps from the Norton current source would have to flow through the 0.8 Ω Norton resistance, producing the exact same voltage, 11.2 volts (E=IR). Thus, we can say that the Thevenin voltage is equal to the Norton current times the Norton resistance:

$$E_{Thevenin} = I_{Norton}R_{Norton}$$

So, if we wanted to convert a Norton equivalent circuit to a Thevenin equivalent circuit, we could use the same resistance and calculate the Thevenin voltage with Ohm's Law.

Conversely, both Thevenin and Norton equivalent circuits should generate the same amount of current through a short circuit across the load terminals. With the Norton equivalent, the short-circuit current would be exactly equal to the Norton source current, which is 14 amps in this case. With the Thevenin equivalent, all 11.2 volts would be applied across the 0.8 Ω Thevenin resistance, producing the exact same current through the short, 14 amps (I=E/R). Thus, we can say that the Norton current is equal to the Thevenin voltage divided by the Thevenin resistance:

$$I_{Norton} = \frac{E_{Thevenin}}{R_{Thevenin}}$$

This equivalence between Thevenin and Norton circuits can be a useful tool in itself, as we shall see in the next section.

REVIEW:

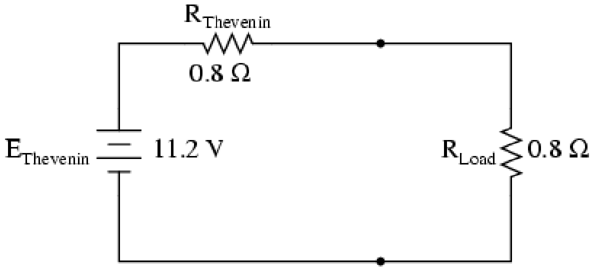
- Thevenin and Norton resistances are equal.
- Thevenin voltage is equal to Norton current times Norton resistance.
- Norton current is equal to Thevenin voltage divided by Thevenin resistance.

Maximum Power Transfer Theorem

The Maximum Power Transfer Theorem is not so much a means of analysis as it is an aid to system design. Simply stated, the maximum amount of power will be dissipated by a load resistance when that load resistance is equal to the Thevenin/Norton resistance of the network supplying the power. If the load resistance is lower or higher than the Thevenin/Norton resistance of the source network, its dissipated power will be less than maximum.

This is essentially what is aimed for in radio transmitter design , where the antenna or transmission line “impedance” is matched to final power amplifier “impedance” for maximum radio frequency power output. Impedance, the overall opposition to AC and DC current, is very similar to resistance, and must be equal between source and load for the greatest amount of power to be transferred to the load. A load impedance that is too high will result in low power output. A load impedance that is too low will not only result in low power output, but possibly overheating of the amplifier due to the power dissipated in its internal (Thevenin or Norton) impedance.

Taking our Thevenin equivalent example circuit, the Maximum Power Transfer Theorem tells us that the load resistance resulting in greatest power dissipation is equal in value to the Thevenin resistance (in this case, 0.8 Ω):



With this value of load resistance, the dissipated power will be 39.2 watts:

	$R_{Thevenin}$	R_{Load}	Total	
E	5.6	5.6	11.2	Volts
I	7	7	7	Amps
R	0.8	0.8	1.6	Ohms
P	39.2	**39.2**	78.4	Watts

If we were to try a lower value for the load resistance (0.5 Ω instead of 0.8 Ω, for example), our power dissipated by the load resistance would decrease:

	$R_{Thevenin}$	R_{Load}	Total	
E	6.892	4.308	11.2	Volts
I	8.615	8.615	8.615	Amps
R	0.8	0.5	1.3	Ohms
P	59.38	**37.11**	96.49	Watts

Power dissipation increased for both the Thevenin resistance and the total circuit, but it decreased for the load resistor. Likewise, if we increase the load resistance (1.1 Ω instead of 0.8 Ω, for example), power dissipation will also be less than it was at 0.8 Ω exactly:

	$R_{Thevenin}$	R_{Load}	Total	
E	4.716	6.484	11.2	Volts
I	5.895	5.895	5.895	Amps
R	0.8	1.1	1.9	Ohms
P	27.80	**38.22**	66.02	Watts

If you were designing a circuit for maximum power dissipation at the load resistance, this theorem would be very useful. Having reduced a network down to a Thevenin voltage and resistance (or Norton current and resistance), you simply set the load resistance equal to that Thevenin or Norton equivalent (or vice versa) to ensure maximum power dissipation at the load. Practical applications of this might include radio transmitter final amplifier stage design (seeking to maximize power delivered to the antenna or transmission line), a *grid tied inverter* loading a solar array, or electric vehicle design (seeking to maximize power delivered to drive motor).

The Maximum Power Transfer Theorem is not: Maximum power transfer does not coincide with maximum efficiency. Application of The Maximum Power Transfer theorem to AC power distribution will not result in maximum or even high efficiency. The goal of high efficiency is more important for AC power distribution, which dictates a relatively low generator impedance compared to load impedance.
Similar to AC power distribution, high fidelity audio amplifiers are designed for a relatively low output impedance and a relatively high speaker load impedance. As a ratio, "output impdance" : "load impedance" is known as *damping factor*, typically in the range of 100 to 1000.
Maximum power transfer does not coincide with the goal of lowest noise. For example, the low-level radio frequency amplifier between the antenna and a radio receiver is often designed for lowest possible noise. This often requires a mismatch of the amplifier input impedance to the antenna as compared with that dictated by the maximum power transfer theorem.

REVIEW:

- The *Maximum Power Transfer Theorem* states that the maximum amount of power will be dissipated by a load resistance if it is equal to the Thevenin or Norton resistance of the network supplying power.
- The Maximum Power Transfer Theorem does **not** satisfy the goal of maximum efficiency.

UNIT-IV
THREE PHASE CIRCUITS

Advantages of Three-phase Circuits

Smooth flow of power (instantaneous power is constant).

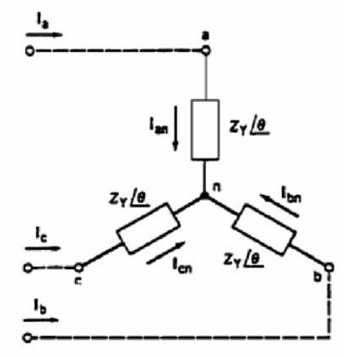
• Constant torque (reduced vibrations).

• The power delivery capacity tripled (increased by 200%) by increasing the number of conductors from 2 to 3 (increased by 50%).

Three-phase Circuits

Wye-Connected System

- The neutral point is grounded
- The three-phase voltages have equal magnitude.
- The phase-shift between the voltages is 120 degrees.



$$V_{an} = |V| \angle 0° = V$$

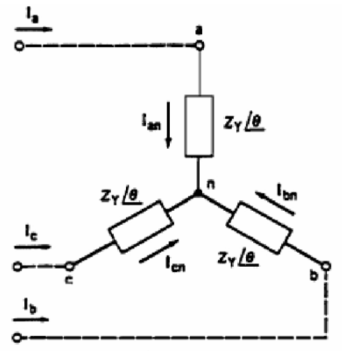
$$V_{bn} = |V| \angle -120°$$

$$V_{cn} = |V| \angle -240°$$

Three-phase Circuits

Wye-Connected System

- Line-to-line voltages are the difference of the phase voltages



$$V_{ab} = V_{an} - V_{bn} = \sqrt{3}\, V \angle 30$$

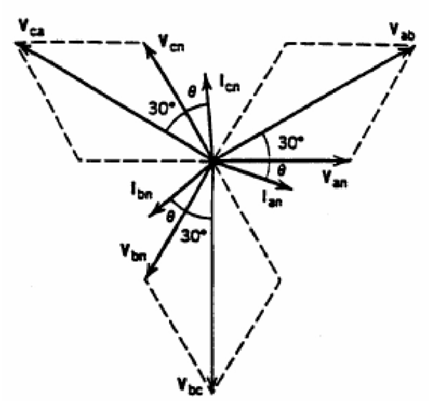
$$V_{bc} = V_{bn} - V_{cn} = \sqrt{3}\, V \angle -90$$

$$V_{ca} = V_{cn} - V_{an} = \sqrt{3}\, V \angle 150$$

Three-phase Circuits

Wye-Connected System

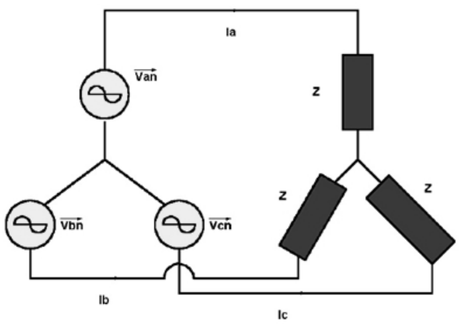
- Phasor diagram is used to visualize the system voltages
- Wye system has two type of voltages: Line-to-neutral, and line-to-line.
- The line-to-neutral voltages are shifted with 120o
- The line-to-line voltage leads the line to neutral voltage with 30o
- The line-to-line voltage is $\sqrt{3}$ times the line-to-neutral voltage



Three-phase Circuits

Wye-Connected Loaded System

- The load is a balanced load and each one = Z
- Each phase voltage drives current through the load.
- The phase current expressions are:

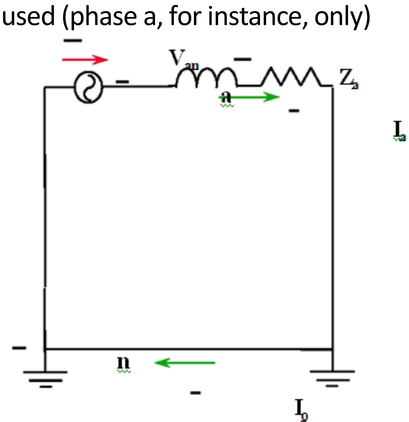


$$I_a = \frac{V_{an}}{Z}, I_b = \frac{V_{bn}}{Z}, I_c = \frac{V_{cn}}{Z}$$

Three-phase Circuits

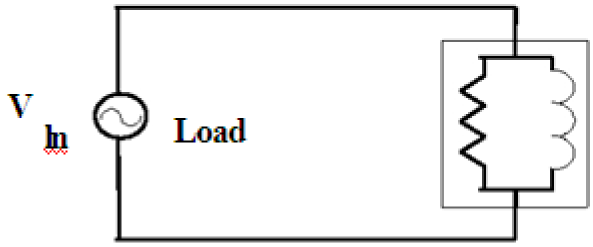
Wye-Connected Loaded System

- Since the load is balanced (Za = Zb = Zc) then: Neutral current = 0
- This case single phase equivalent circuit can be used (phase a, for instance, only)
- Phase b and c are eliminated



Three-phase Circuits
Wye-Connected System with balanced load

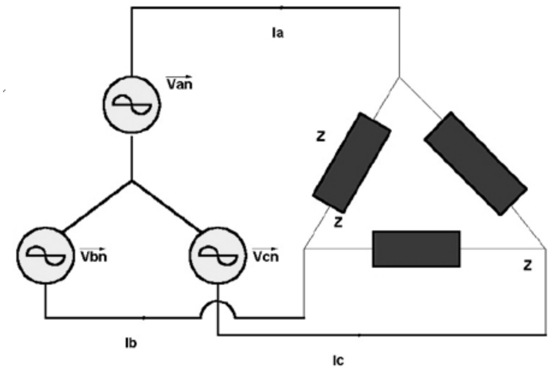
- A single-phase equivalent circuit is used
- Only phase a is drawn, because the magnitude of currents and voltages are the same in each phase. Only the phase angles are different (-120° phase shift)
- The supply voltage is the line to neutral voltage.
- The single phase loads are connected to neutral or ground



Three-phase Circuits
Balanced Delta-Connected System

- The system has only one voltage : the line-to-line voltage (VLL)
- The system has two currents :
    - line current
    - phase current
- The phase currents are:

$$I_a = \frac{V_{ab}}{z}$$



UNIT-III
TRANSIENT ANALYSIS

The two types of voltages and currents we studied are called as AC and DC .The analysis that have been studied so far is called as the steady stable analysis. For a D.C supply, the voltage source or current source is constant throughout and currents and voltages in all the branches also remain constant throughout and hence is called as a steady state.Similar in the case of A.C supply, even though the source is alternating the branch currents and voltages are constant throughout because the amplitude and frequency of this is constant. Hence they are also related to as steady state analysis. Before the circuit reaches the steady state it passes through a state called as the transient state.

In a network containing energy storage elements with the charge in excitation, the currents and voltages charge from one state to other state. The behavior of voltage or current when it is changed from one state to another state is called the transient state. The time taken for the circuit to change from one state to another state is called the transient time.

The transient changes are assumed to occur at time t=0 and represented by a switch.The conditions just before the switch is operated will be designated as i(0-),v(0^-) and conditions after the switch is operated as i(0t) and v(ot). After closing the switch new currents and voltages may appear in the network as the result of initial capacitor voltages the initial inductor currents are because of the network of the current and voltage sources which are introduced

The transient changes may occur due to

1 A change in the interconnections within the network
2 A change in the element values
3 A change in the nature of the excitation itself

The currents changes are assigned to occur at time t=0 and represented by a switch
The switch may be supposed to close or open at time t=0
The switch s is assumed to be an ideal switch which makes or breaks the circuit in zero time interval

1. T=0 is defined as an instant prior to t=0
2. T=0 is defined as the instant immediately after switch

Depending on the circuit parameters, the response changes from an initial steady state to the fire steady state in a time period known as transient period

The response is known as the transient response or transients

To differentiate between immediately before and immediately after the operation of the switch we use – and + signs
The conditions existing just befor the switch is operated or designed as i(0),v(0-) and the conditions after the switch is operated as i(0+) and v(0+)

INITIAL CONDITIONS:

1 resistor –for an ideal resistor we have v(t)=ri(t)
That is the voltage across the resistor will change instantaneously if the current through it changes instantaneously and increases

INDUCTOR: The current cannot change instantaneously in a system of constant inductance. An inductor will not allow any sudden changes in the current throughout it.
If a switch is closed so as ato connect an indicator to a source of energy it will not allow any current to flow at the initial instant and the will act as if it were an open circuit of voltage at the terminals.
If a current of I(0) flows in the conductor at the instant of switching the current will continue to flow and the inductor can be through out as a current source of I(0) amperes.

CAPACITOR: The capacitor will not allow any sudden changes in the voltage across it.

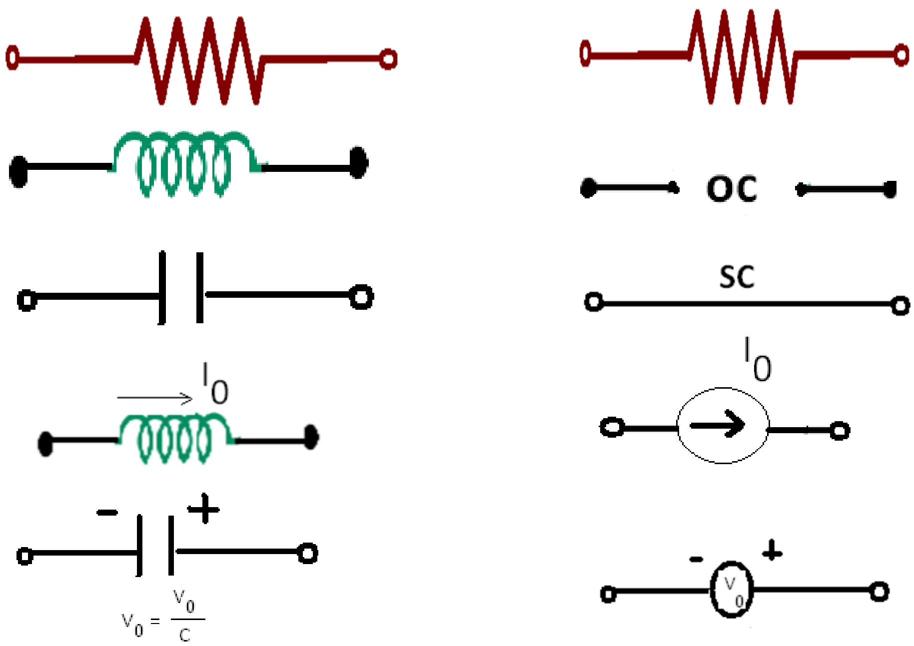
If an unchanged capacitor is connected to an energy source then the voltage across it is initially is zero and hence it acts as shortcircuit.

If an initial change is already existing that is V(0)=Q(0)/C then the capacitor is equivalent to the voltage source of voltage value V(0)=Q(0)/C ;

Where Q(0)=IS INITIAL CHANGE

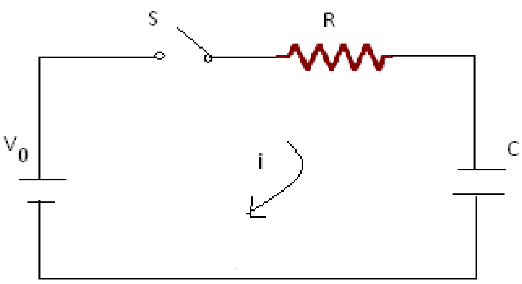
ELEMENT EQUIVALENT CIRCUIT

(At t=0^+)



#In the given circuit above , the switch 'S' closed at t = 0 .

Find the value of i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at t = 0^+, given that V_0 = 10 V, R = 100 Ω and C = 1μf



$V_0 = R_i + \frac{1}{c}\int idt$ --------(1)

At t = 0^+, capacitor accts as a

∴ Short circuit

$$\therefore\ i(t) = \frac{V}{R} = \frac{10}{100} = 0.1\ A$$

Differentiate (1)

$0 = R\frac{di}{dt} + \frac{i}{c}$ $\quad ------(2)$

$\frac{di}{dt} = \frac{-i}{cR}$

At t = 0^+, $\frac{di}{dt}(ot) = \frac{-i^{i(o^+)}}{cR}$

$= \frac{-0.1}{10^{-6}+100} = -10^{-3}\ amp/sec$

Differentiate (2) = $\frac{Rd^2i}{dt^2} + \frac{1}{C}\frac{di}{dt}$

$\frac{d^2i}{dt^2} = \frac{-i}{cR}\frac{di}{dt}$ at t = 0^+, $\frac{d^2i}{dt^2}(0^+) = \frac{-i}{cR}\frac{di}{dt}\ (0^+)$

$$= -\frac{1}{10^{-6}X100}X\ 10^{-3}$$

\# In the circuit shown, V = 10V, R = 10 Ω L = 1 Hand C =1 μF and $V_C(0) = 0$ find i(0^+) $\frac{di}{dt}$ (O^+) and $\frac{d^2i}{dt^2}(0^+)$

$$V = L\frac{di}{dt} + Ri + \frac{i}{c}\int idt$$

At t = (0^+) , the inductor acts as open circuit and the capacitor as short circuit.

$$\therefore i(0^+) = 0$$

$$V = L\frac{di}{dt} + R(0) + 0$$

$$\therefore \frac{di}{dt}(0^+) = \frac{V}{L} = \frac{10}{1} = 10A/\sec \text{ Type equation here.}$$

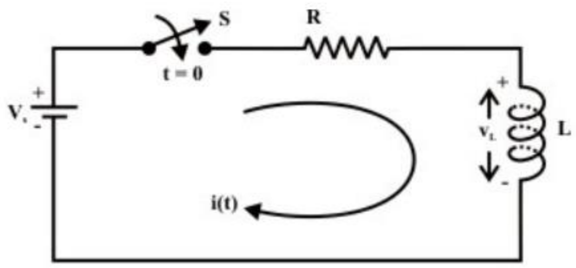
$$0 = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} = 0$$

$\frac{d^2i}{dt^2}(0^+) = -\frac{R}{L}\frac{di}{dt}(0^+) + \frac{i(0^+)}{C}$

$$= \frac{-10}{1}X10+0$$

$$= -100 \text{ Amp/sec}$$

DC RESPONSE OF A SERIES RL CIRCUIT:



$$V = Ri + L\frac{di}{dt}$$

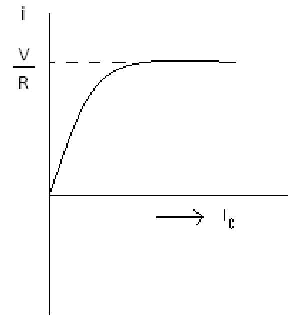
$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$$

$$i = e^{(\frac{-R}{L})t}\frac{V}{L}e^{(\frac{-R}{L})t} + Ce^{(\frac{-R}{L})t}$$

$$i = \frac{V}{L} X \frac{L}{R} e^{(\frac{-R}{L})t} e^{(\frac{R}{L})t} + Ce^{(\frac{R}{L})t}$$

$$I = \frac{V}{R} + Ce^{(\frac{-R}{L})t}$$

At t = 0 I = 0



$$0 = \frac{V}{R} + Ce^{0} = C = -\frac{V}{R}$$

$$i = \frac{V}{R} - \frac{V}{R} e^{(\frac{-R}{L})t}$$

$$V_R = iR = V - V\,e^{\left(\frac{-R}{L}\right)^{t}}$$

$$V_L = L\frac{di}{dt} = Ve^{\left(\frac{-R}{L}\right)^{t}}$$

$$V = V_R + V_L$$

The current i(t) increases exponentially starting from zero to final value $\frac{V}{R}$

Time constant

It is defined as the time in seconds at which exponent of the exponential term is unity.

$$\frac{R}{L}t = 1$$

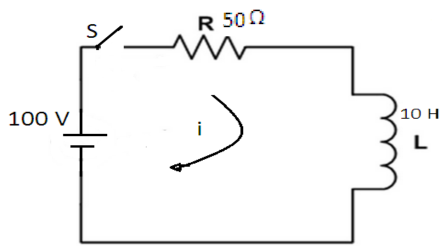
$$t = \frac{L}{R} = time\ constant\ .$$

A siries RL circuit, with R = 50Ω and L = 10H has a constant voltage V = 100V applied at t = 0 by closing of a switch.

Find a) The equation for I , V_R and V_L

b) The current at t = 0.5 seconds

c) The time at which $V_R = V_L$



$$100 = 50i + 10\frac{di}{dt}$$

$$\frac{di}{dt} + 5i = 10$$

$$i = i_{PI} + I_{cp}$$

$i = 2 + C\, e^{-5t}$

At t = 0 i = 0,

$C = -2$

$i = 2 - 2e^{-5t}$

$V_R = R_i = 100(1 - e^{-5t})$

$V_L = L\frac{di}{dt} = 100\, e^{-5t}$

b) put t=0.5

$i = 2[1 - 2e^{-5t}] = 2\,[\,1 - e^{-5X0.5}]$

$=2[1 - 0.082] = 1.836A$

C) When $V_R = V_L = 50$ V

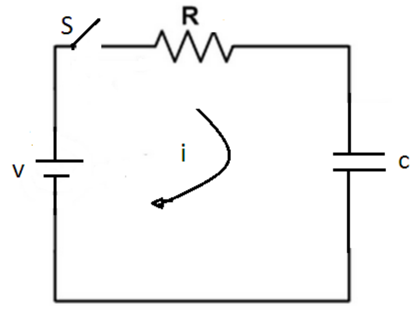
$N_L = 100\, e^{-5t} = 50$

$e^{-5t} = 0.5$

$5t = 0.693$

$T = 0.13819$

<u>DC RESPONSE OF SERIES RC CIRCUIT:</u>



$V = Ri + \frac{1}{C}\int idt$

$0 = R\frac{di}{dt} + \frac{1}{C}$

$$\frac{di}{dt} + \frac{1}{RC}i = 0$$

i= $e^{-\frac{1}{RC}} \int 0\, e^{-\frac{1}{RC}}dt + C_1\, e^{-\frac{1}{RC}t}$

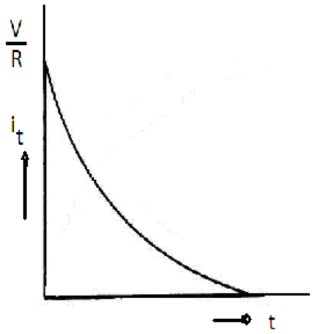
$$i = C_1 e^{-\frac{1}{RC}t}$$

At t=0 , I = $e^{-\frac{1}{RC}t}$

$$\frac{V}{R} = C_1 e^{-\frac{1}{RC}t}$$

$$C_1 e^0 = \frac{V}{R}$$

$C_1 = \frac{V}{R}$ $\quad\quad i = \frac{V}{R} e^{-\frac{1}{RC}t}$



V_R = Ri

= R$\frac{V}{R}$ $e^{\left[-\frac{1}{Rt}\right]t}$

= V $e^{\left[-\frac{1}{Rt}\right]t}$

V_C = $\frac{1}{c}\int idt$

= $\frac{1}{c}\int \frac{V}{R} e^{\left[-\frac{1}{Rt}\right]t} dt$

= $\frac{1}{c}\int \frac{V}{R} e^{-\frac{1}{Rt}t} XRC$

V_C = V$e^{-\frac{1}{Rt}t}$+C

At t = 0 $\quad$ V_C = 0

V_C = V

V_C = V - V$e^{-\frac{1}{Rt}t}$

VC = V(1 - $e^{-\frac{1}{Rt}t}$)

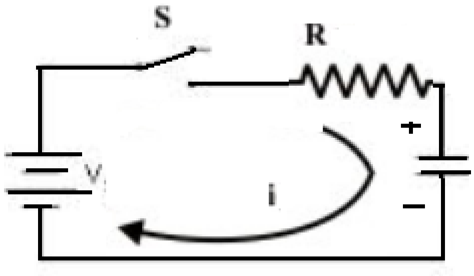
Time constant:

It is the time in seconds at which the exponent of exponential term is unity.

$$\frac{t}{RC} = 0$$

T = RC = TC

SERIES RC CIRCUIT:



V = Ri + $\frac{1}{C}\int idt$

$i = \frac{dq}{dt}$

V = $\frac{dq}{dt} + \frac{q}{RC} = \frac{V}{R}$

$$\frac{V}{RS} = \int Q(S) - q(0) + \frac{Q(s)}{RS}$$

q(0⁻) = 0

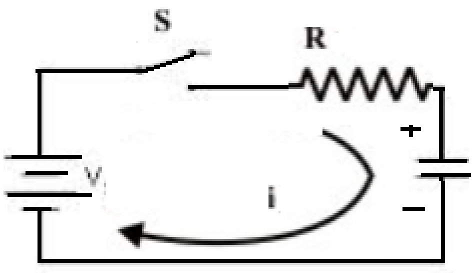
$\frac{V}{RS} = Q(1)\left[S + \frac{1}{RC}\right]$ $\Longrightarrow$ Q(s) = $\frac{\frac{V}{RS}}{S+\frac{1}{RC}} = \frac{\frac{V}{R}}{S\left[S+\frac{1}{RC}\right]}$

Q(s) = $\frac{A}{S} + \frac{B}{S+\frac{1}{RC}}$

Q(s) = $\frac{CV}{S} - \frac{CV}{S+\frac{1}{RC}}$

Q(t) = CV – CV $e^{\left(-\frac{1}{RC}\right)t}$

SERIES RC CIRCUIT:



V = Ri + $\frac{1}{C}\int idt$

$$\frac{V}{S} = RI(s) + \frac{1}{C}\left[\frac{I(S)}{S} + \frac{I^{-1}(0^+)}{S}\right]$$

$$I^{-1}(0^+) = \int idt = it = q(0)$$

$$\frac{V}{S} = RI(s) + \frac{1}{C}\left[\frac{I(S)}{S} + \frac{q(0)}{S}\right]$$

$= RI(s) + \left[\frac{I(S)}{CS} + \frac{q(0)}{S}\right]$

$$RI(s) + \left[\frac{I(S)}{CS} + \frac{V(0)}{S}\right]$$

$$\frac{V}{S} = I(S)\left[R + \frac{1}{CS}\right]$$

DC RESPONSE OF A SERIES RL CIRCUIT:

V = Ri + L$\frac{di}{dt}$

$$\frac{V}{S} = rI(S) + L\left[\int I(S) - i(0)\right]$$

$$RI(s) + S\,L\,I\,(S) = \frac{V}{S}$$

$$I(S) = \frac{V}{\int(R + SL)} = \frac{V}{S}$$

$$I(S) = \frac{V}{\int(R + SL)} = \frac{\frac{V}{L}}{S\left[S + \frac{R}{L}\right]}$$

S = 0 , $\frac{A\left(S+\frac{R}{L}\right)+BI}{S\left(S+\frac{R}{L}\right)}$ = $\frac{\frac{V}{L}}{S\left[S+\frac{R}{L}\right]}$

A+B = 0

AR/L = V/L

A = V/R B =-V/R

I(t) = $\frac{V}{R} - \frac{V}{R}e^{-\left(\frac{R}{L}\right)t}$

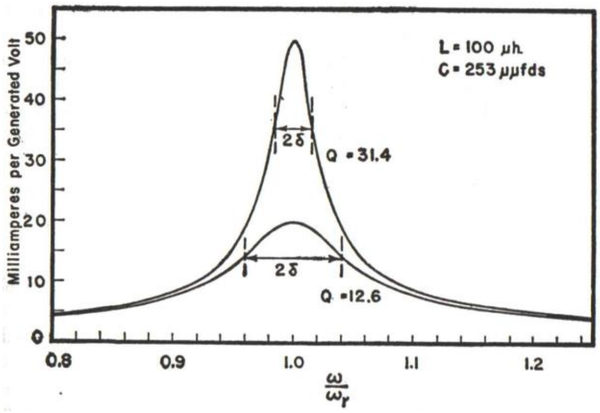
II(S) = $\frac{V}{RS} - \frac{V}{R\left[S+\frac{R}{L}\right]}$

Unit-IV

FILTERS

Filters -Resonant circuits

- Resonant circuits will select relatively narrow bands of frequencies and reject others.
- Reactive networks are available that will freely pass desired band of frequencies while almost suppressing other bands of frequencies.
- Such reactive networks are called filters.



Ideal Filter

An ideal filter will pass all frequencies in a given band without (attenuation) reduction in magnitude, and totally suppress all other frequencies. Such an ideal performance is not possible but can be approached with complex design.

Filter circuits are widely used and vary in complexity from relatively simple power supply filter of a.c. operated radio receiver to complex filter sets used to separate the various voice channels in carrier frequency telephone circuits.

Application of Filter circuit

Whenever alternating currents occupying different frequency bands are to be separated, filter circuits have an application.

Neper - Decibel

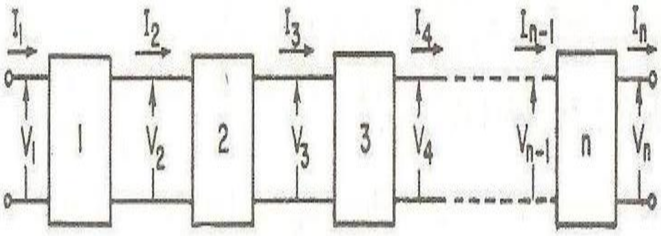
In filter circuits the performance of the circuit is expressed in terms of ratio of input –current to output-current magnitude.

$$Performance = \frac{|Input \text{ current}|}{|\text{Output current}|}$$

If the ratios of voltage to current at input and output of the network are equal then

$$\left|\frac{I_1}{I_2}\right| = \left|\frac{V_1}{V_2}\right| \text{...............}(1)$$

If several networks are used in cascade as shown if figure the overall performance will become



$$\left|\frac{V_1}{V_2}\right| X \left|\frac{V_2}{V_3}\right| X \left|\frac{V_3}{V_4}\right| X \text{.....} \left|\frac{V_{n-1}}{V_n}\right| = \left|\frac{V_1}{V_n}\right| \text{----------------}(2)$$

Which may also me stated as

$$A_1\angle\alpha.A_2\angle\beta.1A_3\angle\gamma.A_4\angle\delta = A_1A_2A_3A_4\angle\alpha+\beta+\gamma+\delta$$

Both the processes employing multiplication of magnitudes. In general the process of addition or subtraction may be carried out with greater ease than the process of multiplication and division. It is therefore of interest to note that

$$e^{\alpha}\times e^{b}\times e^{c}\times.....e^{n} = e^{a+b+c+....+n}$$

is an application in which addition is substituted for multiplication.

If the voltage ratios are defined as

$$\left|\frac{V_1}{V_2}\right| = e^{a};\left|\frac{V_2}{V_3}\right| = e^{b};\left|\frac{V_3}{V_4}\right| = e^{c};.......etc$$

Eq. (2) becomes

$$\left|\frac{V_1}{V_n}\right| = e^{a+b+c+........+n}$$

If the natural logarithm (ln) of both sides is taken, then

$$\ln\left|\frac{V_1}{V_2}\right| = a+b+c+d.........+n \text{ ----------------(3)}$$

Consequently if the ratio of each individual network is given as " n " to an exponent, the logarithm of the current or voltage ratios for all the networks in series is very easily obtained as the simple sum of the various exponents. It has become common, for this reason, to define

$$\left|\frac{V_1}{V_2}\right| = \left|\frac{I_1}{I_2}\right| = e^{N} \text{ ------------(4)}$$

under condition of equal impedance associated with input and output circuits
The unit of "N" has been given the name nepers and defined as

$$N \text{ nepers} = \ln\left|\frac{V_1}{V_2}\right| = \ln\left|\frac{I_1}{I_2}\right| \text{ ……………..(5)}$$

Two voltages, or two currents, differ by one neper when one of them is "e" times as large as the other

Obviously, ratios of input to output power may also may also be expressed In this fashion. That is,

$$\left|\frac{P_1}{P_2}\right| = e^{2N}$$

The number of nepers represents a convenient measure of power loss or power gain of a network.
Loses or gains of successive networks then may be introduced by addition or subtraction of their appropriate N values.

" bel " - " decibel "

The telephone industry proposed and has popularized a similar unit based on logarithm to the base 10, naming the unit " bel " for Alexander Graham Bell

The "bel" is defined as the logarithm of a power ratio,

number of bels = $\log \frac{P_1}{P_2}$

It has been found that a unit one-tenth as large is more convenient, and the smaller unit is called the decibel, abbreviated "db" , defined as

$$dB = 10 \log \frac{P_1}{P_2} \quad \text{................(6)}$$

In case of equal impedance in input and output circuits,

$$dB = 20 \log \frac{I_1}{I_2} = 20 \log \frac{V_1}{V_2} \quad \text{..................(7)}$$

Equating the values for the power ratios,

$$e^{2N} = 10^{\frac{dB}{10}}$$

Taking logarithm on both sides

8.686 N = dB

or 1 neper = 8.686 dB

Is obtained as the relation between nepers and decibel.

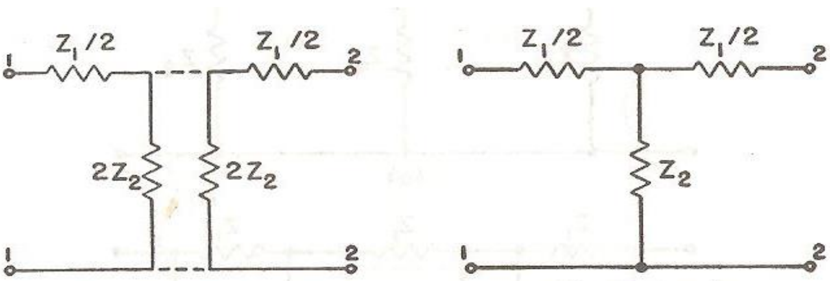
The ears hear sound intensities on a logarithmically and not on a linear one.

Characteristic impedance of symmetrical networks

Symmetrical T section Network

For symmetrical network the series arms of T network are equal

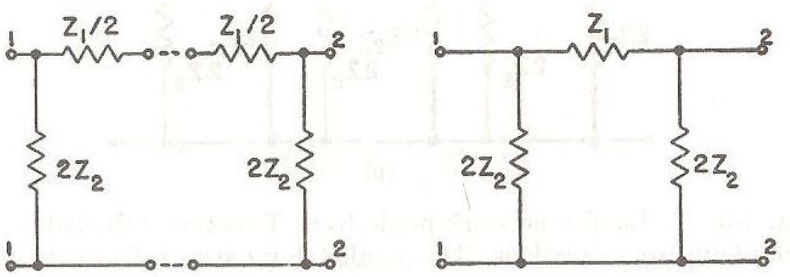
$$Z_1 = Z_2 = \frac{Z_1}{2}$$



(a) Two L-Sections (b) T-Section

Symmetrical π Network

$$Z_a = Z_c = 2Z_2$$

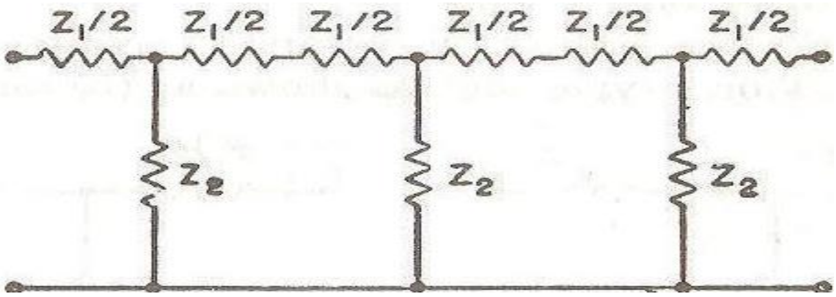


(c) Two L-Sections (d) Π - Section

Both T and π networks can be considered as built of unsymmetrical L half sections, connected together in one fashion for T and oppositely for the π network.

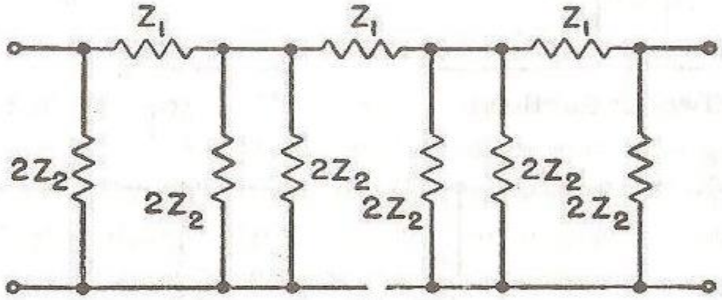
A series connection of several T or π networks leads to so-called "ladder networks" which are indistinguishable one from the other except for the end or terminating L half section as shown.

Ladder Network made from T section



Ladder Network built from π section

The parallel shunt arms will be combined

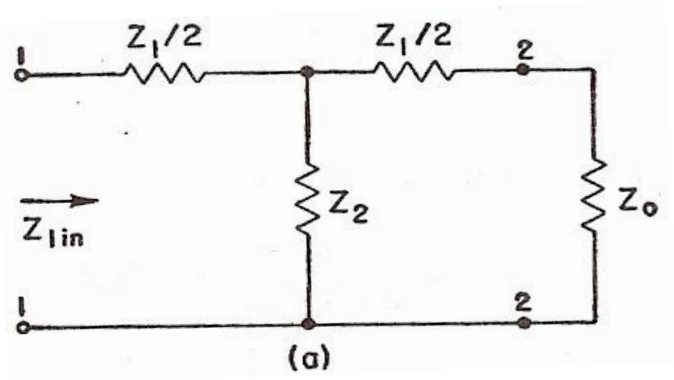


For a symmetrical network the image impedance Z_{1i} and Z_{2i} are equal to each other and the image impedance is then called characteristic impedance or iterative impedance, Z_0.

That is , if a symmetrical T network is terminated in, its input impedance will also be Z_0 , or the impedance transformation ration is unity. Z_0

The term iterative impedance is apparent if the terminating impedance Z_0 is considered as the input impedance of a chain of similar networks in which case Z_0 is iterated at the input to each network.

Characteristic Impedance of Symmetrical T section network



For T Network terminated in Z0

$$Z_{1in} = \frac{Z_1}{2} + \frac{Z_2(Z_1/2 + Z_0)}{Z_1/2 + Z_2 + Z_0}$$

When $Z_{1in} = Z_0$

$$Z_0 = \frac{Z_1^2/4 + Z_1 Z_2 + Z_2 Z_0 + Z_1 Z_0/2}{Z_1/2 + Z_2 + Z_0}$$

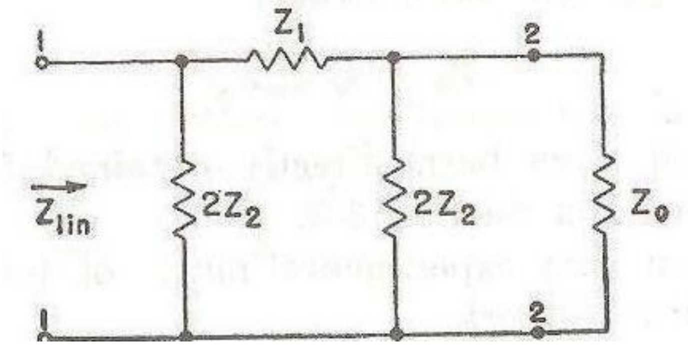
$$Z_0^2 = \frac{Z_1^2}{4} + Z_1 Z_2$$

Characteristic Impedance
for a symmetrical T section

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2} = \sqrt{Z_1 Z_2 (1 + \frac{Z_1}{4Z_2}}$$

Characteristic impedance Z_0 is that impedance, if it terminates a symmetrical network, its input impedance will also be Z_0
Z_0 is fully decided by the network's intrinsic properties, such as physical dimensions and electrical properties of network.

Characteristic Impedance π section Z_0



$$Z_{1in} = \frac{\left[Z_1 + (\frac{2Z_2 Z_0}{2Z_2 + Z_0}) \right] 2Z_2}{Z_1 + \frac{2Z_2 Z_0}{2Z_2 + Z_0 + 2Z_2}}$$

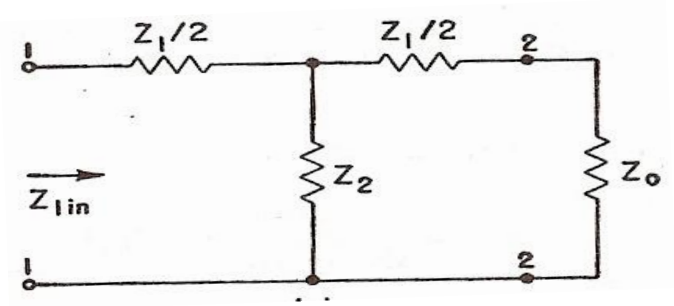
When $Z_{1in} = Z_0$, for symmetrical π

Characteristic Impedance $Z_{0\pi} = \sqrt{\frac{Z_1 Z_2}{1 + Z_1 / 4Z_2}}$

$$Z_{1oc} = Z_{oc} = \frac{Z_1}{2} + Z_2$$

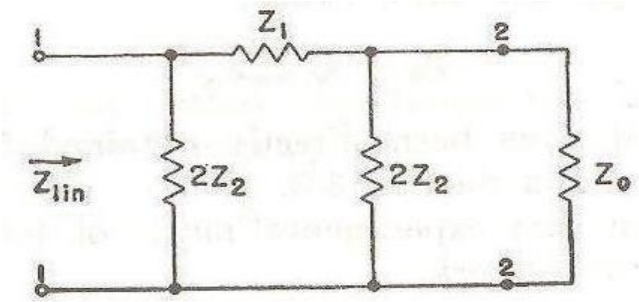
$$Z_{1sc} = Z_{sc} = \frac{Z_1}{2} + \frac{Z_1 Z_2 / 2}{Z_1/2 + Z_2}$$

$$Z_{oc} Z_{sc} = \frac{Z_1^2}{4} + Z_1 Z_2 = Z_{0T}^2$$



$$Z_{0c} Z_{sc} = \frac{4Z_2^2 Z_1}{Z_1 + 4Z_2} = Z_{0\pi}^2$$

$$Z_0 = \sqrt{Z_{oc} Z_{sc}}$$



Propagation constant γ

Under the assumption of equal input and output impedances, which may be Z_0, for a symmetrical network, the current ratio.

The magnitude ratio does not express the complete network performance, the phase angle between the currents being needed as well. The use of exponential can be extended to include the phasor current ratio if it be defined that under the condition of Z_0

$$\frac{I_1}{I_2} = e^{\gamma}$$

Where is a complex number defined by $\gamma = \alpha + j\beta$

Hence $\frac{I_1}{I_2} = e^{\gamma} = e^{\alpha + j\beta}$

If $\frac{I_1}{I_2} = A\angle\beta$

$A = \left|\frac{I_1}{I_2}\right| = e^{\alpha} \quad \beta = e^{j\beta}$

With Z_0 termination, it is also true,

$$\left|\frac{V_1}{V_2}\right| = e^{\gamma}$$

The term has been given the name propagation constant

α = attenuation constant, it determines the magnitude ratio between input and output quantities.
= It is the attenuation produced in passing the network.

Units of attenuation is nepers

β = phase constant. It determines the phase angle between input and output quantities.
= the phase shift introduced by the network.
= The delay undergone by the signal as it passes through the network.
= If phase shift occurs, it indicate the propagation of signal through the network.

The unit of phase shift is radians.

If a number of sections all having a common Z_0
the ratio of currents is

$$\frac{I_1}{I_2} \times \frac{I_2}{I_3} \times \frac{I_3}{I_4} \times \text{........} = \frac{I_1}{I_n}$$

from which

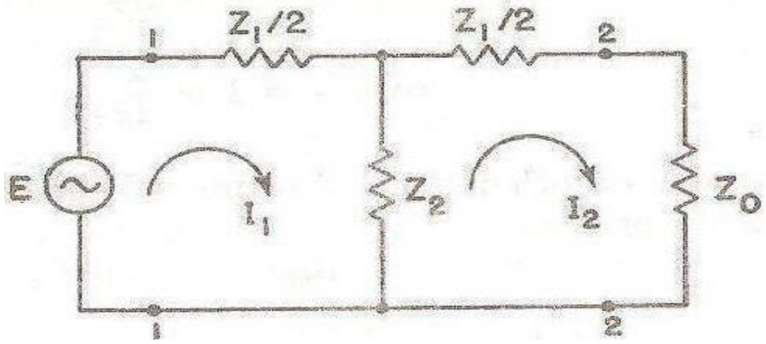
$$e^{\gamma 1} \times e^{\gamma 2} \times e^{\gamma 3} \times \text{........} = e^{\gamma n}$$

and taking the natural logarithm,

$$\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \text{..................} = \gamma_n$$

Thus the overall propagation constant is equal to the sum of the individual propagation constants.

Physical properties of symmetrical networks



Use the definition of γ and the introduction of e^{γ} as the ratio of current for a Z_0 termination leads to useful results

The T network in figure is considered equivalent to any connected symmetrical network terminated in a Z_0 termination. From the mesh equations the current ratio can be shown as

$$\frac{I_1}{I_2} = \frac{Z_1/2 + Z_2 + Z_0}{Z_2} = e^{\gamma}$$

Where the characteristic impedance is given as

$$Z_0^2 = \frac{Z_1^2}{4} + Z_1 Z_2$$

The propagation constant γ can be related to the network parameters as follows:

$$e^{\gamma} = 1 + 2\frac{Z_1}{2Z_2} + \sqrt{(\frac{Z_1}{2Z_2})^2 + \frac{Z_1}{Z_2}}$$

$$\gamma = \ln\left[1 + \frac{Z_1}{2Z_2} + \sqrt{(\frac{Z_1}{2Z_2})^2 + \frac{Z_1}{Z_2}}\right]$$

$$Z_{in} = Z_{11} - \frac{Z_{12}^2}{Z_{22}}$$

$$Z_{in} = Z_0\left(\frac{Z_R \cosh\ \gamma\ +\ Z_0\ \sinh\ \gamma}{Z_0\ \cosh\ \gamma + Z_R\ \sinh\ \gamma}\right)$$

For short circuit, $Z_R = 0$

$$Z_{SC} = Z_0\ \tanh\ \gamma$$

For a open circuit $Z_R = \infty$

$$Z_{\lim\ Z\to\infty} = \frac{Z_0}{\tanh\ \gamma}$$

From these these two equations it can be shown that

$$\tanh\ \gamma = \sqrt{\frac{Z_{SC}}{Z_{OC}}}$$

$$Z_0 = \sqrt{Z_{OC} Z_{SC}}$$

Thus the propagation constant γ and the characteristic impedance Z_0 can be evaluated using measurable parameters

$$Z_{SC} and Z_{OC}$$

Filter fundamentals

Pass band – Stop band:

The propagation constant is $\gamma = \alpha + j\beta$

For $\alpha = 0$ or $I_1 = I_2$

There is no attenuation, only phase shift occurs.

It is pass band.

when $\alpha\ =\ +$ ve; $I_1 < I_2$, attenuation occurs;

-Stop band

γ Is conveniently studied by use of the expression.

$$\sinh\ \frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}}$$

It is assumed that the network contains only pure reactance and thus $Z_1/4Z_2$ will be real and either positive or negative, depending on the type of reactance used for Z_1 and Z_2

Expanding the above expression

$$\sinh\frac{\gamma}{2} = \sinh\left(\frac{\alpha}{2} + \frac{j\beta}{2}\right)$$

$$= \sinh\frac{\alpha}{2}\cos\frac{\beta}{2} + j\cosh\frac{\alpha}{2}\sin\frac{\beta}{2}$$

It contains much information

If Z_1 and Z_2 *are* the same type reactances then

$$\left|\frac{Z_1}{4Z_2}\right| > 0 \text{ or the ratio is positive and real.}$$

This condition implies a stop or attenuation band of frequencies.

The attenuation will be given by

$$\alpha = 2\sinh^{-1}\sqrt{\frac{Z_1}{4Z_2}}$$

If Z_1 and Z_2 *are* opposite type reactances then

$$\left|\frac{Z_1}{4Z_2}\right| < 0 \text{ or the radical is imaginary.}$$

This results in the following conclusion for pass band.

$$-1 < \frac{Z_1}{4Z_2} < 0$$

The phase angle in this pass band will be given by

$$\beta = 2\sin^{-1}\sqrt{\frac{Z_1}{4Z_2}}$$

Another condition for stop band is given as follows:

$$\text{when } \frac{Z_1}{4Z_2} < -1$$

$$\left|\frac{Z_1}{4Z_2}\right| > 0 \quad \text{Stop band.}$$

$$-1 < \frac{Z_1}{4Z_2} < 0 \quad \text{pass band}$$

$$\frac{Z_1}{4Z_2} < -1 \quad \text{stop band}$$

Cut-off frequency

The frequency at which the network changes from pass band to stop band, or vice versa, are called cut-off frequencies.

These frequencies occur when

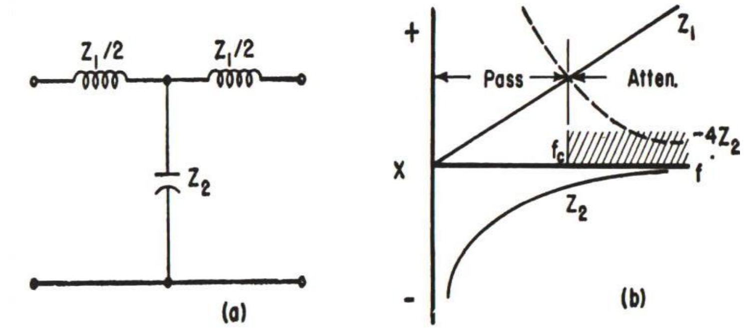
$$\frac{Z_1}{4Z_2} = 0 \quad \text{or} \quad Z_1 = 0$$

$$\frac{Z_1}{4Z_2} = -1 \quad \text{or} \quad Z_1 = -4Z_2 \text{.....................(48)}$$

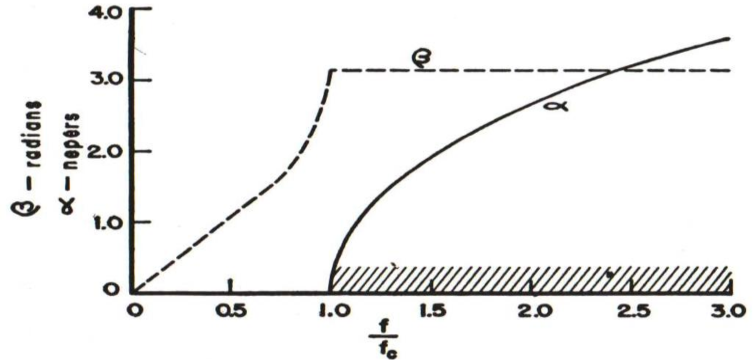
where Z_1 & Z_2 are opposite types of reactances.

Since Z_1 and Z_2 may have number of combinations, as L and C elements, or as parallel and series combinations, a variety of types of performance are possible.

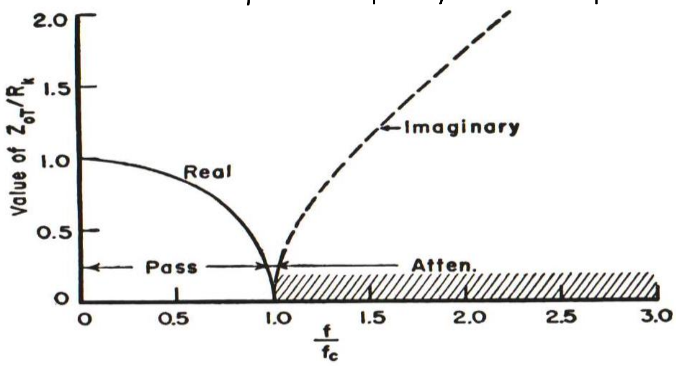
Constant k- type low pass filter



(a) Low pass filter section; (b) reactance curves demonstrating that (a) is a low pass section or has pass band between Z1 = 0 and Z1 = - 4 Z2

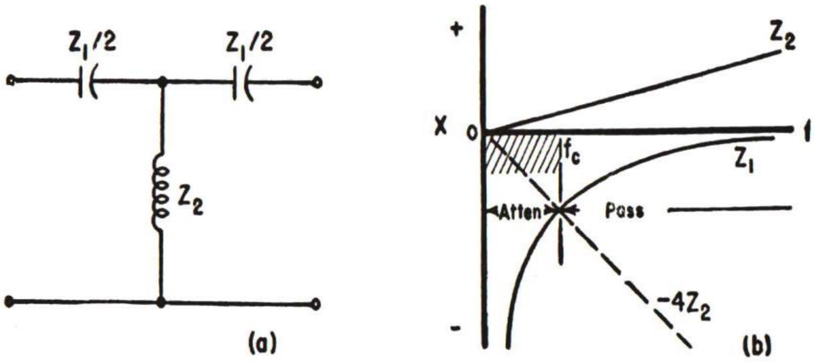


Variation of α and β with frequency for the low pass filter



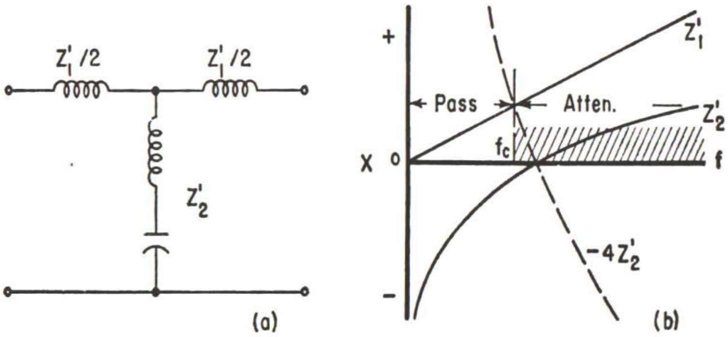
Variation of $\frac{Z_{OT}}{R_k}$ with frequency for low pass filter.

Constant k high pass filter

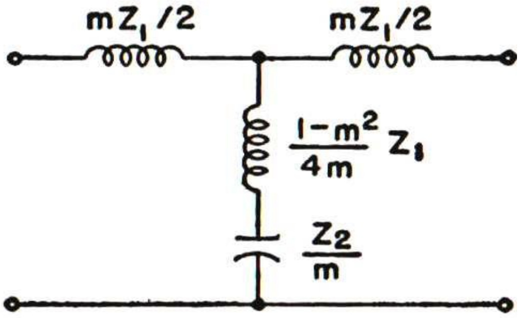


(a) High pass filter;(b) Reactance curves demonstrating that (a) is a high pass filter or pass band between Z1 = 0 and Z1 = - 4Z2

m-derived T section

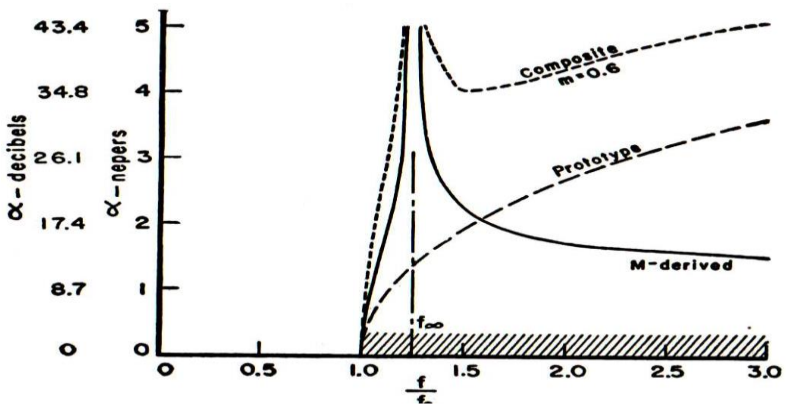


a) Derivation of a low pass section having a sharp cut-off section (b) reactance curves for (a)

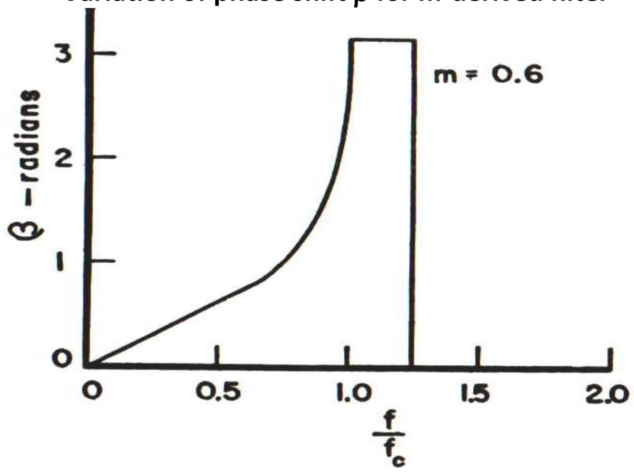


m-derived low pass filter

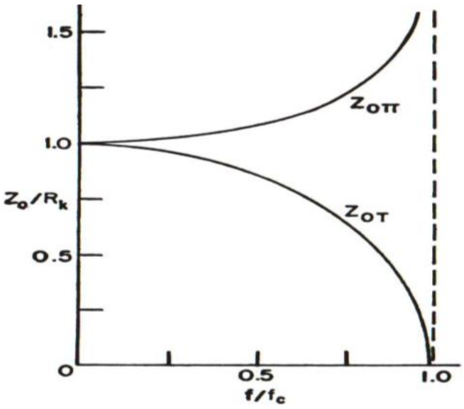
Variation of attenuation for the prototype amd m-derived sections and the composite result of two in series.



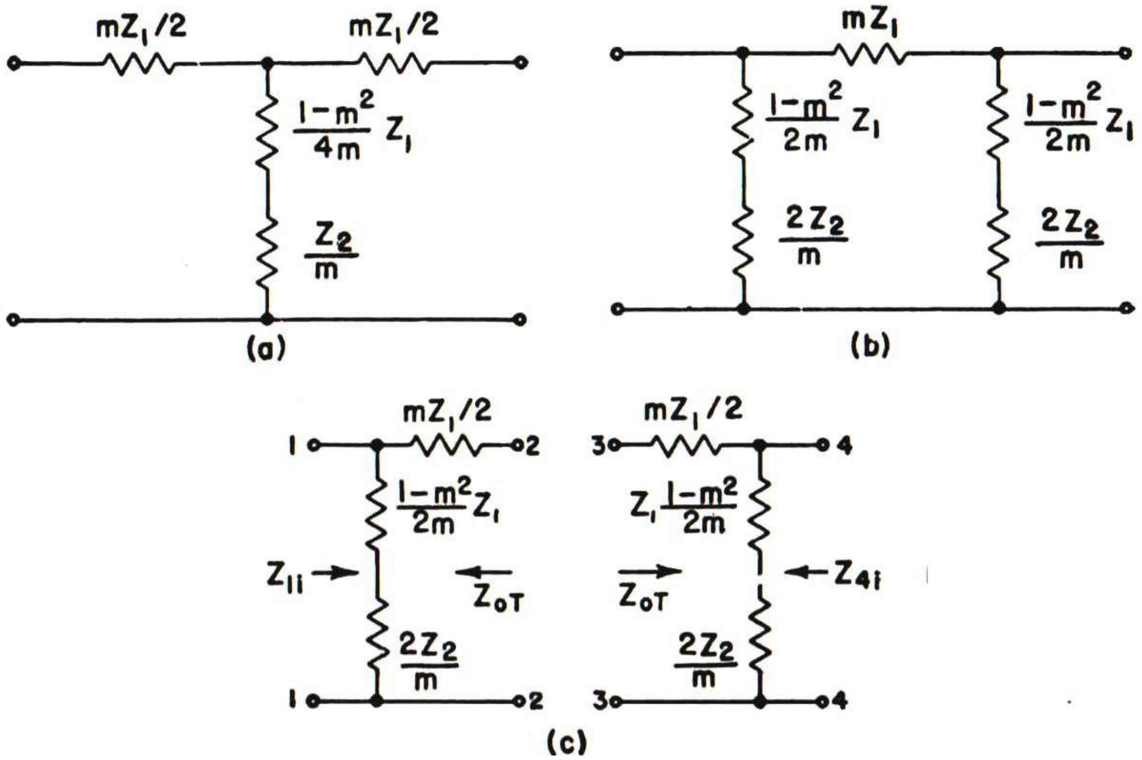
Variation of phase shift β for m-derived filter



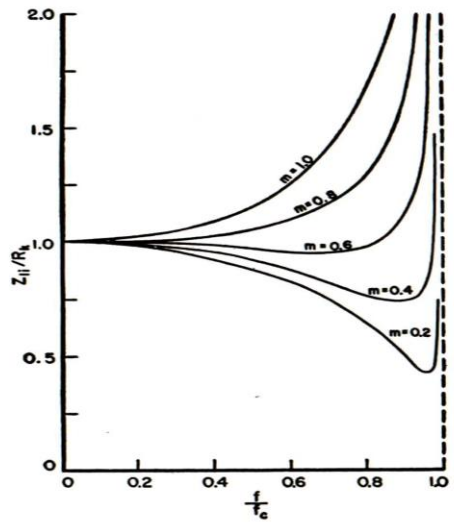
Variation of Z_0 over the pass band for T and π networks



(a) m-derived T section; (b) π section formed by rearranging of (a); © circuit of (b) split into L sections.



Variation of Z1 of the L section over the pass band plotted for various m valus



LOCUS DIAGRAMS

Locus diagrams are the graphical representations of the way in which the response of electrical circuits vary, when one or more parameters are continuously changing. They help us to study the way in which

a. Current / power factor vary, when voltage is kept constant,
b. Voltage / power factor vary, when current is kept constant, when one of the parameters of the circuit (whether series or parallel) is varied.

The Locus diagrams yield such important information as I_{max}, I_{min} , V_{max} ,V_{min} & the power factor`s at which they occur. In some parallel circuits, they will also indicate whether or not, a condition for response is possible.

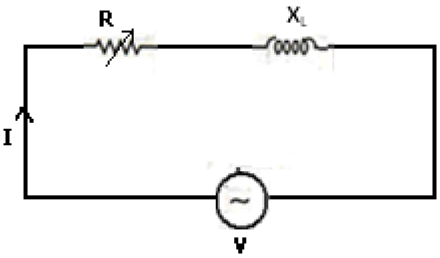
RL Series Circuit:

Consider an R – X_L series circuit as shown below, across which a constant voltage is applied. By varying R or X_L, a wide range of currents and potential differences can be obtained.

`R ` can be varied by the rheostatic adjustment and X_L can be varied by using a variable inductor or by applying a variable frequency source.

When the variations are uniform and lie between 0 and infinity, the resulting locus diagrams are circles

Case 1: when `R` is varied



When R = 0 , the current is maximum and is given by $I_{max} = \frac{V}{X_L}$ and lags V by 90^0

∴ Power factor is zero

When R = infinity, the current is minimum and is given by I_{min} = 0, $\emptyset = 0$ and power factor = 1

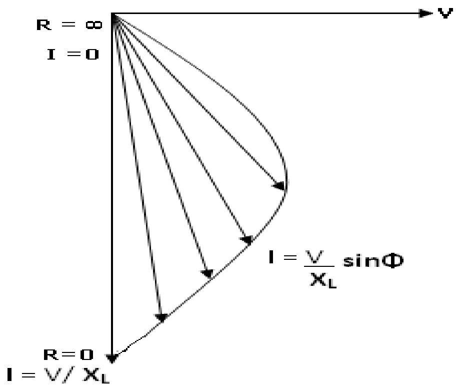
For any other values of`R`, the current lags the voltage by an angle $\emptyset = \tan^{-1}\frac{X_L}{R}$

∴ The general expression for current is

$$I = \frac{V}{\sqrt{R^2 + {X_L}^2}} = \frac{V}{Z}\frac{X_L}{X_L} = \frac{V}{X_L}\frac{X_L}{Z} = \frac{V}{X_L}\sin\emptyset$$

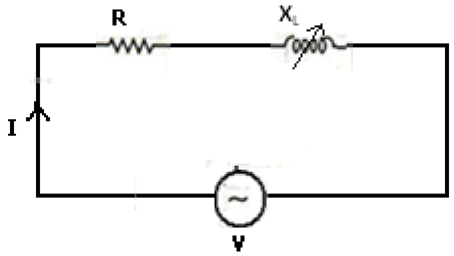
The equation $I = \frac{V}{X_L}\sin\emptyset$ is the equation of a circle in the polar form, where $\frac{V}{X_L}$ is the diameter of the circle.

The Locus diagram of current i.e the way in which the current varies in the circuit, as `R` is varied from zero to infinity is shown in below which is a semi -circle.



∴ Locus of current in a series RL circuit is a semi circuit with radius = $\frac{V}{2X_L}$ & whose center is given by $(0, \frac{V}{2X_L})$

Case 2: When X_L is varied



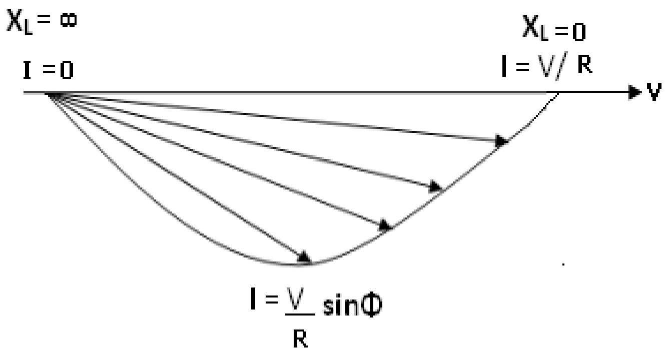
When $X_L = 0$, current is maximum and is given by $\frac{V}{R}$ and is in phase with V. The power factor is unity.

When X_L = to infinity, the current is zero, the power factor is zero and $\emptyset = 90^0$

For any other value of \`R\`, the current lags the voltage by an angle $\emptyset = \frac{X_L}{R}$

∴ The general expression for current is $I = \frac{V}{\sqrt{R^2+X_L{}^2}} = \frac{V}{R}\frac{R}{R} = \frac{V\,R}{R\,Z} = \frac{V}{R}\cos\,\emptyset$

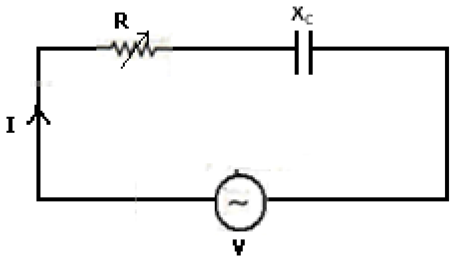
The equation of a circle in the polar form where $\frac{V}{R}$ is the diameter of the circle



∴ The Locus of current in a series RL circuit is a semi circuit whose radius is $\frac{V}{2R}$ and whose center is $\left(\frac{V}{2R}, 0\right)$

RC Series Circuit:

Case 1: when \`R\` is varied



When R = 0 current is maximum and is given by $I_{max} = \frac{V}{X_c}$, which leads the voltage by 90^0 .Power factor is zero.

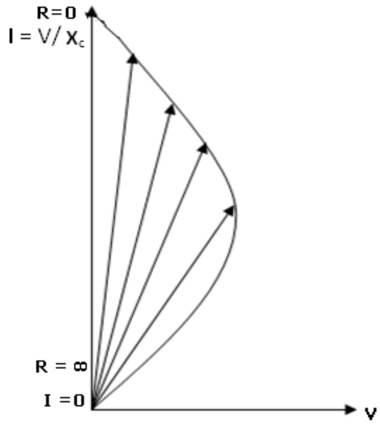
When R = ∞, the current is zero. The power factor is unity & $\emptyset = 0$

For any other value of R the current leads the voltage by an angle $\emptyset = tan^{-1}\frac{X_c}{R}$

∴ The general expression for current is

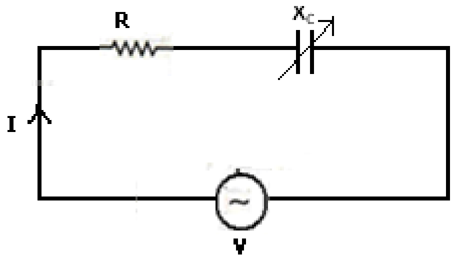
$$I = \frac{V}{\sqrt{R^2 + X_L{}^2}} = \frac{V}{Z}\frac{X_c}{X_c} = \frac{V}{X_c}\ \frac{X_c}{Z} = \frac{V}{X_c}\sin\emptyset$$

$\therefore \frac{V}{X_c}\sin\emptyset$ is the equation of a circle in the polar form, where $\frac{V}{X_c}$ is the diameter of the circle.



$\therefore$ Locus is a semi – circle where radius is $\frac{R}{2X_c}$ & center is $\left(0,\ \frac{V}{2X_c}\right)$.

Case 2: Where X_c is varied



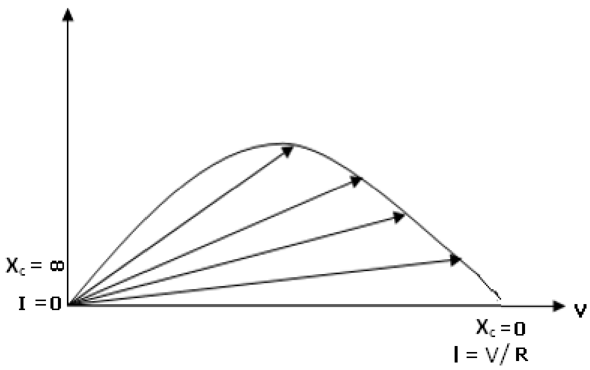
When X_c = 0 , current is maximum & is given by $I_{max} = \frac{V}{R}$, which is in phase with V. Power factor is unity and $\emptyset = 0$

When $X_c = \infty$, the current is zero. Power factor is 0 & $\emptyset = 90^0$, for any other value of X_c , the current leads the voltage by an angle $\emptyset = tan^{-1}\frac{X_c}{R}$

The general equation for the current is

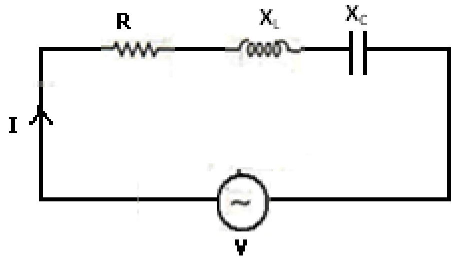
$$I = \frac{V}{Z} = \frac{V}{Z}\frac{R}{R} = \frac{V}{R}X\frac{R}{Z} = \frac{V}{R}\ cos\emptyset$$

The equation $I = \frac{V}{R}\ cos\emptyset$ is the equation of the circle in polar form, where $\frac{V}{R}$ is the diameter of the circle.



$\therefore$ The locus is a circle of radius $\left(\frac{V}{2R}\ ,0\right)$.

RLC series circuit:



The figure represents an $R - X_L - X_c$ series circuit across which, a constant voltage source is applied `I` is the current flowing through the circuit. The characteristics of this circuit can be studied by varying any one of the parameters, R, X_L, X_c & f .

Case1: when R is varied and the other three parameters are constant, the locus diagram of current are similar to those of

a) $R - X_L$ series circuit, if $X_L > X_c$

b) $R - X_c$ Series circuit if $X_c > X_L$

The only difference would be, the resulting reactance is either $X_L - X_c$ or $X_c - X_L$

Case2: When X_L is varied

When $X_c = 0$ the circuit behaves as an R-X_c series circuit & the current is given by

$$I = \frac{V}{\sqrt{R^2 + X_L{}^2}} \quad \& \quad \emptyset = tan^{-1}\frac{X_c}{R}$$

When $X_L = X_c$, the circuit behaves as a pure resistance, circuit the current is maximum or is given by $I_{max} = \frac{V}{R}$ & $\emptyset = 0$ The power factor is unity

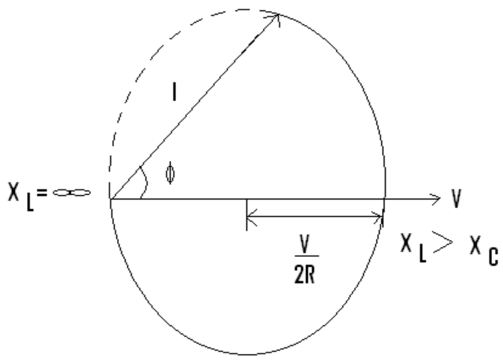
Where $X_L > X_C$, The circuit behaves as an $R - X_L$ series circuit & the current is given by

$$I = \frac{V}{\sqrt{R^2+(X_L-X_C)^2}} \;\&\; \emptyset = tan^{-1}\frac{X_L-X_c}{R} \;(lagging)$$

When $X_L = \infty$, I = 0

For any value of X_L Lying between X_C & ∞, the locus of current is a semi circle of radius = $\frac{V}{2R}$.

The complete locus diagram of current as X_L varies from zero to infinity is as shown below.



Case3 : When X_C is varied

When $X_C = 0$ the circuit behaves as an R-X_L series circuit & the circuit is given by

$$I = \frac{V}{\sqrt{R^2+(X_L-X_C)^2}} \;\&\; \emptyset = tan^{-1}\frac{X_L}{R} \;(lagging)$$

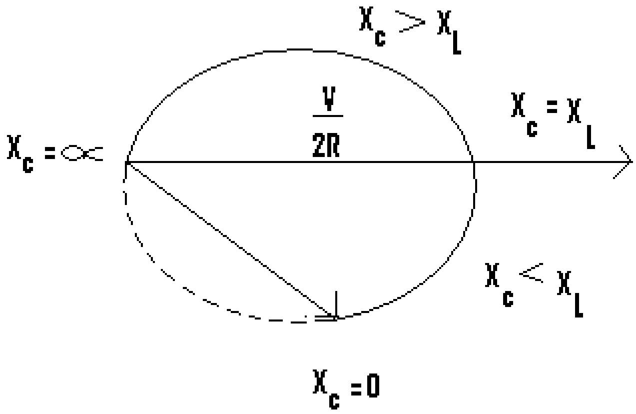
When $X_C = X_L$, the circuit behaves as a pure resistance circuit. The current is maximum and is given by $I_{max} = \frac{V}{R}$ & $\emptyset = 0$. The power factor is unity

When $X_C > X_L$, the circuit behaves as an $R - X_C$ series circuit and the current is given by

I = $\frac{V}{\sqrt{R^2+(X_C-X_L)^2}}$ & $\emptyset = tan^{-1}\frac{X_C-X_L}{R}$ $(leading)$

For any value of X_C lying between X_L & ∞, the locus of current is a semi circle of radius $\frac{V}{2R}$.

The complete locus diagram of current as X_C varies from o to ∞ is as shown below.



Case 4: When 'f' is varied

When f=0, $X_c = \infty$, hence I=0.

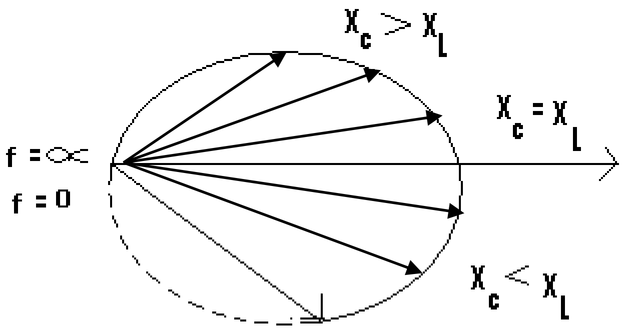
For values of 'f', for which X_c>X_L, the circuit behaves as an R-X_c series circuit and the locus is a semi circle in upper half of X-Y plane with V/2R as radius.

For values of 'f', which X_c=X_L, the current is maximum and is equal to I_{max}= V/R, Φ=0, P.F=1.

For values of 'f', for which X_c <X_L the circuit behaves as an R-X_L series circuit and the locus is a semi circle in lower half of X-Y plane with V/2R as radius.

For f = ∞, X_L= ∞, X_c =0 and hence I=0.

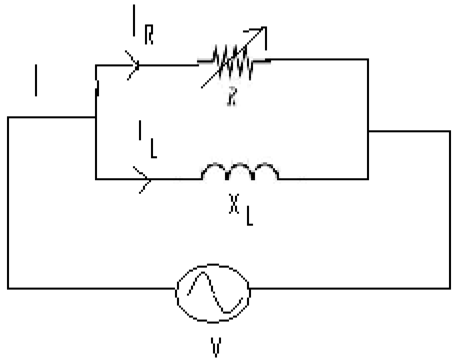
Therefore the complete locus diagram of current as f varies from 0 to ∞ is as shown in figure bellow



Locus Diagrams of parallel circuits:

When a constant voltage, constant frequency source is applied across a parallel circuit and any one parameters in one of the parallel branches is verified, current varies only in that branch and the total current locus is get by adding the variable current locus with the constant current flowing in the other branch.

Case 1: R & X_L in parallel R Varying:



Consider a parallel circuit as shown below, across which a constant voltage, constant frequency source is applied.

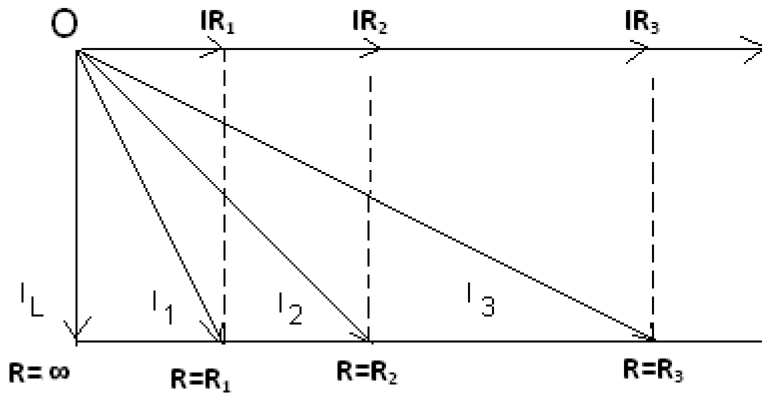
$$\vec{I} = \vec{I_L} + \vec{I_R}$$

As X_L Is constant I_L is constant

As R is variable I_R is Variable

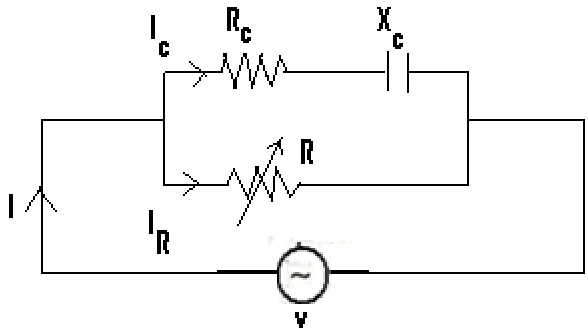
When R = ∞, I_R = 0 and I = I_L which lags V by 90^0

For any other values of R = R_1, the current I_L remains constant, but $I_{R1} = \frac{V}{R_1}$ and is in phase with V.



For other values of R=R_2, R_3.. etc., I_{R2},I_{R3} etc., and I_1 ,I_2 etc., can be found and plotted.

Case 2: R-X_C in parallel with R & 'R' varying.



Consider a parallel circuit consisting of R_C-X_C branch in parallel with 'R' as shown.

$$I = \vec{I_C} + \vec{I_R}$$

As R_C & X_C are constants, I_C remains constant & is given by

$$I_C = \frac{V}{\sqrt{(R_C{}^2)+(X_C{}^2)}} \quad \& \quad \emptyset_C = \tan^{-1}\frac{X_C}{R}\ (leading)$$

As R is variable I_R is also variable.

When R = ∞ I_R = 0, hence I = I_C

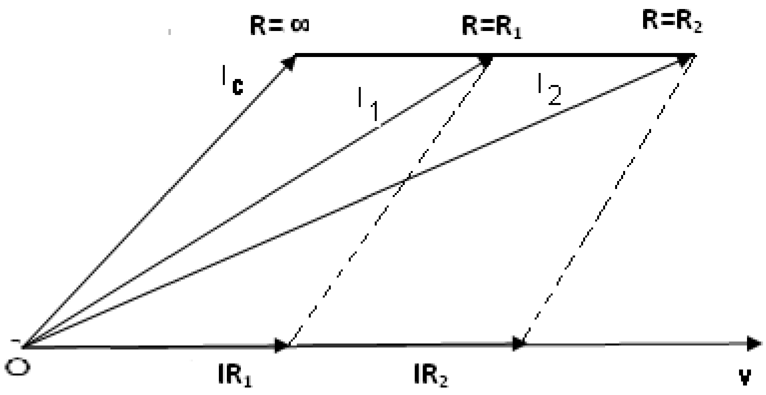
For any other values of R = R_1, I_C remains constant, but $I_{R1} = \frac{V}{R_1}$ & is in phase with V

The total current is given by Type equation here.

$$\vec{I} = \vec{I_C} + \vec{I_R}$$

Similarly for other values of $R_2, R_3, etc., I_{R_2}, I_{R_3} etc., \& I_2\, I_3 etc.,$ can be plotted

The locus of the total current is as shown below.



\# A 230 volts, 50 H source is connected to a series circuit consisting of a resistance of 30 ohms and an inductance which varies between 0.03 henries and 0.15 henries. Draw the Locus Diagram of current.

Diameter of circle = $\frac{V}{R} = \frac{230}{30}$ = 7.67 amps

X_{min} = 2X 3.14 X 50 X 0.03 = 9.42 ohms

$$I_{max} = \frac{230}{\sqrt{(30)^2+(9.42)}}$$

X_{max} = 2X3.14x50x0.15 = 47.1ohms

$$I_{min} = \frac{230}{\sqrt{(30)^2+(47.1)^2}} = 4.52 \text{ am}$$

FOURIER ANALYSIS OF AC CIRCUITS

Fourier Theorem

- Fourier Theorem provides a method for representing discontinuous functions by a trigonometric series. Fourier series are series of cosine and sine terms and are used to represent general periodic signals.

- According to Fourier Theorem, any periodic and non- sinusoidal function can be expressed as an infinite sum of sine and cosine functions that are integral multiples of fundamental frequency ω.

- Fourier developed a technique of analyzing non- sinusoidal waveforms applicable to a wide range to engineering problems. Later it was applied to analyze electric circuits excited by non-sinusoidal waveforms. Fourier analysis deals with Fourier series and Fourier transforms and has several applications in mathematics, science and engineering particularly in the area of communications and signal processing.

Trigonometric Series

Periodic signals can be represented by sum of sinusoids whose frequencies are harmonics or integer multiples of fundamental frequency. The Fourier series representation of a periodic signal will be of the form

$$X(t) = \mathbf{a_0 + a_1\cos\omega t + a_2\cos 2\omega t + \text{----} + b_1\sin\omega t + b_2\sin 2\omega t + \text{-----}}$$

Where a_0 a_1, a_2 ----- b_1, b_2 ------are real constants

Such a series is called a trigonometric series and a_n and b_n are called the coefficient of the series

$$X(t) = a_0 + \sum_{n=1}^{\infty}(a_n cosn\omega t + b_n sinn\omega t) \qquad \text{------(1)}$$

Determination of constant a_0

Integrating on both sides of equation (1) from –π/ω to π/ω

$$\int_{-\pi/\omega}^{\pi/\omega} x(t)\,dt = \int_{-\pi/\omega}^{\pi/\omega} [a_0 + \textstyle\sum_{n=1}^{\infty}(a_n cosn\omega t + b_n sinn\omega t)]dt$$

$$= a_0\int_{-\pi/\omega}^{\pi/\omega} dt + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi/\omega}^{\pi/\omega} cosn\omega dt + b_n\int_{-\pi/\omega}^{\pi/\omega} sinn\omega dt\right]$$

The first term on the right equals 2π/ω a_0

All the other integrals on the right are zero

$$\int_0^{2\pi/\omega} x(t)\,dt = \frac{2\pi}{\omega}a_0$$

$$a_0 = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} x(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(t)\,d(\omega t)$$

Determination of constant a_n

$$\int_0^{2\pi/\omega} x(t)\,cosn\omega t = \int_0^{2\pi/\omega} a_0 cosn\omega t\,dt + \int_0^{2\pi/\omega} a_1 cos\omega t\;\; cosn\omega t\,dt \text{ -----}$$
$$+ \int_0^{2\pi/\omega} a_n cos^2 n\omega t\,dt + --- \int_0^{2\pi/\omega} b_1 sin\omega t\; cosn\omega t\,dt + \int_0^{2\pi/\omega} b_2 sin2\omega t\; cosn\omega t\,dt + --$$

The definite integral on the right side zero except

$$\int_0^{2\pi/\omega} a_n \cos^2 n\omega t \, dt \quad = \frac{\pi}{\omega} a_r$$

$$a_n = \frac{\pi}{\omega} \int_0^{2\pi/\omega} x(t) \cos n\omega t \, dt = \frac{1}{\pi} \int_0^{2\pi} x(t) \cos n\omega t \, d(\omega t)$$

Determination of constant b_n

$$\int_0^{2\pi/\omega} x(t) \cos n\omega t \quad b_n = \frac{\pi}{\omega} b_n$$

$$b_n = \frac{\pi}{\omega} \int_0^{2\pi/\omega} x(t) \sin n\omega t \, dt = \frac{1}{\pi} \int_0^{2\pi} x(t) \sin n\omega t \, d(\omega t)$$

Dirichlet Conditions

Any periodic waveform x(t) = x(t+T) can be expressed by Fourier series provided that the following conditions are satisfied.

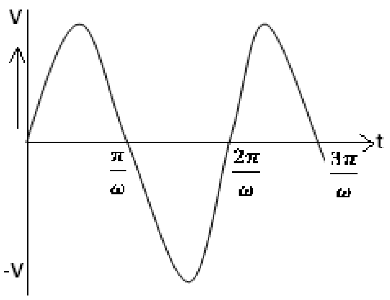
1. The function x(t) has a finite number of discontinuities in one period.
2. The function x(t) is absolutely integral over one period and has a finite average value for the period T.
3. The function x(t) has a finite number of positive and negative maxima in one period.
4. The function x(t) has a finite average value in one period

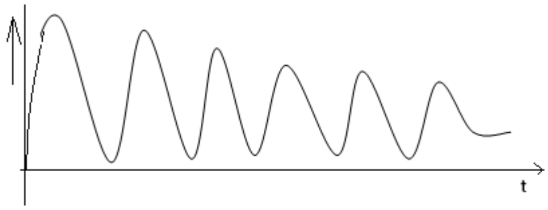
When these conditions called Dirichlet conditions are satisfied the Fourier series exists and can be written in trigonometric form

$f(t) = a_0 + a_1 \cos\omega t + a_2 \cos 2\omega t +$---------$+ b_1 \sin\omega t + b_2 \sin 2\omega t +$----- -----(1)

The Fourier coefficients a's and b's are determined for a given waveform by multiplying both sides of equation (1) by cosnωt (a_n) and by sinnωt and integrating over a full period.

The period of the fundamental, 2π/ω is the period of the series, since each term in the series has a frequency which is an integral multiple of the fundamental.





If a periodic function can be expressed as the sum of a finite or infinite number of sinusoidal functions, the responses of linear networks to non-sinusoidal excitations can be determined by applying the Superposition theorem. The Fourier method provides the means for solving this type of problems.

Exponential Fourier series

If each of the sine and cosine terms in the trigonometric series are expressed by its exponential equivalent, then a series of exponential terms are obtained.

$$f(t) = \frac{a_0}{2} + a_0\left(\frac{e^{j\omega t}+e^{-j\omega t}}{2}\right) + a_2\left(\frac{e^{j2\omega t}+e^{-j2\omega t}}{2}\right) + - - - + b_1\left(\frac{e^{j\omega t}-e^{-j\omega t}}{2j}\right) + b_2\left(\frac{e^{j2\omega t}-e^{-j2\omega t}}{2j}\right) + \quad ----(1)$$

Rearranging

$$f(t) = - - + \left(\frac{a_2}{2} - \frac{b_2}{2j}\right)e^{-j2\omega t} + \left(\frac{a_1}{2} - \frac{b_1}{2j}\right)e^{-j\omega t} + \frac{a_0}{2} + \left(\frac{a_1}{2} + \frac{b_1}{2j}\right)e^{j\omega t} + \left(\frac{a_2}{2} + \frac{b_2}{2j}\right)e^{j2\omega t} + \quad ----(2)$$

We now define a new complex constant A such that

$$A_0 = \frac{1}{2}a_0;\ A_n = \frac{1}{2}(a_n - jb_n),\ A_{-n} = \frac{1}{2}(a_n + jb_n) \quad ----(3)$$

And rewrite (2) as

$$f(t) = \{- - - + A_{-2}e^{-j2\omega t} + A_{-1}e^{-j\omega t} + A_0 + A_1e^{j\omega t} + A_2e^{j2\omega t} + - -\} \quad ----(4)$$

To obtain the evaluation integral for the A_n coefficients, we multiply (4) on both sides by $e^{-jn\omega t}$ and integrate over the full period:

$$\int_0^{2\pi} f(t)\ e^{-jn\omega t}\,d(\omega t) = - - - - - - - + \int_0^{2\pi} A_{-2}e^{-j2\omega t}\,e^{-jn\omega t}d(\omega t) + \int_0^{2\pi} A_{-1}e^{-j\omega t}\,e^{-jn\omega t}d(\omega t) + \int_0^{2\pi} A_0\,e^{-jn\omega t}d(\omega t) + \int_0^{2\pi} A_1e^{j\omega t}\,e^{-jn\omega t}d(\omega t) + - - + \int_0^{2\pi} A_ne^{jn\omega t}\,e^{-jn\omega t}d(\omega t) \quad ----(5)$$

The definite integrals on the right side of (5) are all zero except $\int_0^{2\pi} A_n d(\omega t)$ which has the value $2\pi A_n$ then

$$A_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-jn\omega t}d(\omega t)$$

or with t as the variable

$$A_n = \frac{1}{T}\int_0^{T} f(t)e^{-jn\omega t}dt \quad -----(6)$$

Just as with the $_n$ and b_n evaluation integrals, the limits of integration in (6) need cover any convenient full period and not necessarily 0 to 2π or 0 to T.

The trigonometric series coefficients are derived from the exponential series coefficients as follows first add and them subtract the expressions for A_n and A_{-n} in (3), thus

$A_n + A_{-n} = \frac{1}{2}(a_n - jb_n) + \frac{1}{2}(a_n + jb_n)$

From which

$$\mathbf{a_n = A_n + A_{-n}} \quad -----(7)$$

and

or

$$A_n - A_{-n} = \frac{1}{2}(a_n - jb_n - a_n - jb_n)$$

$$\mathbf{b_n = j(A_n - A_{-n})} \quad -----(8)$$

Line Spectra and Phase Spectra

Fourier coefficient C_n of the exponential form is a complex quantity and cn be represented by

$$\mathbf{C_n = Re\,[C_n] + j\,Im\,[C_n]} \quad -----(1)$$

The real part of C_n is

$$\mathrm{Re}\,[Cn] = \frac{1}{2\pi}\int_0^{2\pi} x(t)\ cosn\omega t \quad -----(2)$$

The imaginary part of C_n is

$$\text{Im}\,[\text{Cn}] = \frac{1}{2\pi}\int_0^{2\pi} x(t)\ \sin n\omega t \qquad \text{-----(3)}$$

Re [C_n] is an even function of n, where as Im [C_n] is an odd function of n. Therefore, the amplitude spectrum of the Fourier series is given by

$$|\text{Cn}| = \{\text{Re}^2[C_n] + \text{Im}^2[C_n]\}^{1/2} \qquad \text{-----(4)}$$

While the phase spectrum is given by

$$\theta_n = \tan^{-1}\left[\frac{\text{Im}[C_n]}{\text{Re}[C_n]}\right] \qquad \text{-----(5)}$$

The Frequency Spectrum

Consider the Fourier series

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n cosm\omega t + b_n sinn\omega t)$$

Let $a_{n=} C_n \cos \alpha_n$

and $b_{n=} C_n \sin \alpha_n$ -----(6)

Therefore

$$x(t) = a_0 + \sum_{n=1}^{\infty}(C_n \cos \alpha_n cosn\omega t + C_n \sin \alpha_n sinn\omega t)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega t - \alpha_n) \qquad \text{-----(7)}$$

From Equation (6) we have

$$tan\,\alpha_n = \frac{b_n}{a_n}$$

$$\alpha_n = tan^{-1}\frac{b_n}{a_n} \qquad \text{-----(8)}$$

Also, we have

$$C_n = \sqrt{a_n{}^2 + b_n{}^2} \qquad \text{-----(9)}$$

The magnitude of C_n is plotted against nω, and the graph obtained is called the frequency spectrum of the given waveform x(t). The amplitudes decrease rapidly for waves with rapidly convergent series. Waves with discontinuities such as sawtooth and square wave have spectra with slowly decreasing amplitudes since their series have strong high harmonics.

Application of Fourier's Series

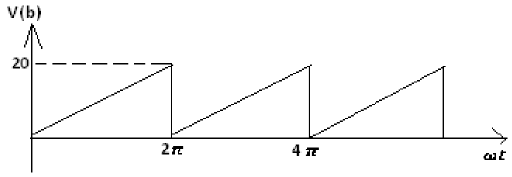
The typical applications of Fourier's series include:

- **Spectrum analysis:** It permits the analysis of a complex waveform. A complex waveform comprises of many harmonics along with the DC component. The Fourier series provides the spectrum of a signal. The spectrum consists of amplitudes and phases of the harmonics and frequency. The Fourier series helps to identify the presence of the various frequency components (spectral lines) to indicate the amount of energy at each frequency in the complex signal.
- **Effective design of filters:** As the Fourier spectra are discrete spectra. So, efficient filter design is possible by blocking the frequency components of a signal that are undesirable and allowing the desirable frequency.

- **Analysis of ac circuits when excited by non-sinusoidal signals:** The analysis of AC circuits becomes easier using PHASOR techniques. The Fourier series expansion allows the application of Phasor techniques in this context.
- **Very useful in analysis of harmonic distortion.**

Problems

1. Find the Fourier series for the waveform shown.



The waveform equation v(t) = $\frac{20}{2\pi}\omega t$

The average value of the waveform

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi}\frac{20}{2\pi}\omega t\, d\omega t = \left[\frac{20}{4\pi^2}\frac{(\omega t)^2}{2}\right]_0^{2\pi} = \frac{20}{2} = 10$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} v(t)\, cos\omega t\, d(\omega t)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{20}{2\pi}\omega t\, cos\omega t\, d(\omega t)$$

$$= \frac{20}{2\pi^2}\left[\frac{\omega t}{n}sinn\omega t + \frac{1}{n^2}\ cosn\omega t\right]_0^{2\pi} = \frac{20}{2\pi^2 n^2}[2\pi n\ sinn2\pi + cosn2\pi - cos0]$$

$$= \frac{20}{2\pi^2 n^2}[1-1] \qquad = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{20}{2\pi}\sin n\omega t\, d(\omega t)$$

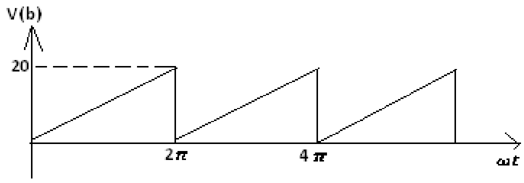
$$= \frac{20}{2\pi^2}\left[-\omega t\frac{cosn\omega t}{n} + \frac{1}{n^2}\sin n\omega t\right]_0^{2\pi} = \frac{-20}{\pi n}$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi}\frac{20}{2\pi}\omega t\, d\omega t$$

$$= \left[20 + \frac{(\omega t)^2 1}{2}\right]_0^{2\pi} = 10$$

$$\therefore v(t) = 10 - \frac{20}{\pi}\sin\omega t - \frac{20}{2\pi}\sin 2\omega t - \frac{20}{3\pi}\sin 3\omega t \quad = 10 - \frac{20}{\pi}\sum_{\pi=1}^{\infty}\frac{sinn\omega t}{\mathrm{n}}$$

2. Find the exponential Fourier series for the waveform shown, using the coefficients of this exponential series obtain a_n and b_n of the trigonometric.



$$A_n = \frac{1}{2\pi}\int_0^{2\pi}\frac{20}{2\pi}\omega t\, e^{-jn\omega t}\, d\omega t$$

$$= \frac{20}{2\pi^2}\left[\frac{e^{-jn\omega t}}{(-jn)^2}(-jn\omega t - 1)\right]_0^{2\pi} = \frac{j20}{2\pi n}$$

$$\therefore x(t) = ---\frac{j20}{4\pi}e^{-j2\omega t} - \frac{j20}{2\pi}e^{-j\omega t} + 10 + \frac{j20}{2\pi}e^{j\omega t} + \frac{j20}{4\pi}e^{j2\omega t} + --$$

$$a_n = A_n + A_{-n} = \left(\frac{j20}{2\pi n} + \frac{j20}{2\pi(-n)}\right) = 0$$

$$b_n = j(A_n + A_{-n}) = j\left(\frac{j20}{2\pi n} - \frac{j20}{2\pi(-n)}\right) = \frac{-20}{\pi n}$$

$$\therefore x(t) = 10 - \frac{20}{\pi}\sin\omega t - \frac{20}{2\pi}\sin 2\omega t - \frac{20}{3\pi}\sin 3\omega - - -$$